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Public Sector and Human Capital: on the Mechanics of Economic Development*

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ABSTRACT

This paper proposes a theory about the allocation of human capital along the development process that helps to understand the controversial impact of this variable in growth regressions. We build a model in which human capital is allocated to three activities: production, tax collection (bureaucracy), and public education. Poor countries have low effective tax rates because tax collection requires human capital, which is scarce. As countries accumulate human capital throughout the transition, the effective tax rate rises, diverting human capital from production to bureaucracy and public education. Consequently, in this stage, human capital has a weak impact on production, even when human capital allocation is efficient. Furthermore, differences in institutional quality may involve a misleading negative correlation between human capital and GDP.

Keywords: Economic Growth, Human Capital, Taxation, Public Education

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1. Introduction

There exists a considerable debate in growth literature over the role of human capital in causing development. Growth theory recognizes the contribution of human capital in the growth process since the seminal contributions of Lucas (1988) and Romer (1990). Furthermore, empirical studies show that the return on education is high at the micro level, especially in developing countries (see Psacharopoulos, 1994; Psacharopoulos and Patrinos, 2004; and Strauss and Duncan, 1995). However, the macroeconomic evidence on the impact of education on growth is controversial. Even when the more recent macro literature has documented a positive correlation between human capital and growth[1], in contrast to the prior papers[2], still the causality relationship between these two variables has been questioned (see Acemoglu and Angrist, 2000; Bils and Klenow, 2000; Bernanke and Rogoff, 2001).

The standard interpretation of the weak causal effect of human capital on development is the inefficient allocation of human capital to unproductive uses, in particular in the public sector (North, 1990; Pritchett, 2001; Mauro, 2004; Blackburn, Bose, and Haque, 2006; Schündeln and Playforth, 2014).

This paper also focuses on how the human capital is allocated among the private and the public sectors during the development process. However, in contrast to previous papers, our explanation is not based on the inefficient allocation of human capital to unproductive activities in the public sector. On the contrary, we consider that public sector activities are productive and essential for development. Establishing a public education system and a modern tax collection structure is necessary for having a successful development process and this requires skilled workers. This absorption of human capital by the public sector at the expense of the private sector in the first stages of development may undermine the growth of the private sector, however, it sets the foundations for creating a modern state and education system needed for development.

We build a model in which human capital has three uses: one in the private sector, to produce goods; and two in the public sector: to collect and manage taxes (we call it "bureaucracy"), and to create human capital throughout the public education system (teachers).

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[1] See Temple (1999), Bond, Hoeffler and Temple (2001), De la Fuente and Doménech (2006), Cohen and Soto (2007), Hanushek and Woessmann (2012), Schoellman (2012), Manuelli and Seshadri (2014), Hendricks and Schoellman (2018), Angrist et al. (2021).

[2] First empirical analyses found a weak and even negative correlation between human capital and growth. See, for instance, Kyriacou (1999), Benhabib and Spiegel (1994), Islam (1995), Nonneman and Vanhoudt (1996), Caselli, Esquivel and Lefort (1996).

There is a statutory tax rate but the tax collection requires skilled workers[3] (bureaucrats). In order to fully implement the statutory tax rate a large enough number of bureaucrats is needed. If this is not the case, the effective tax rate increases with the number of bureaucrats recruited by the government. Taxes finance the public education system, which is essential for producing human capital. These features of the model involve a feedback process: a higher level of human capital implies more bureaucrats, who collect more taxes to finance a public expenditure on education which, in turn, generates more human capital.

Given that skilled workers collect taxes, when countries are poor (human capital is scarce), the effective tax rate is low[4]. However, along the transition, human capital becomes increasingly abundant, involving an increasing effective tax rate and, thus, a rising deviation of human capital from production (private sector) to both bureaucracy and public education (public sector)[5]. As a result, this reallocation of skilled workers implies a low impact of human capital on production. Nevertheless, in contrast with previous literature, this diversion of human capital from the private to the public sector does not mean that human capital is involved in unproductive activities. On the contrary, the public education system plays an essential role in human capital formation, while creating an administration (bureaucracy) is necessary for collecting taxes to finance such a public education system.

Furthermore, we show that the optimal allocation implies that when the initial amount of human capital per capita is lower than the steady-state level, human capital and the effective tax rate rise along the transition. Thus, the efficient portion of human capital devoted to the public sector increases, while the percentage dedicated to production (private sector) decreases. Consequently, the fact that at the earlier stage of development an increasing part of human capital is committed to public sector activities, such as bureaucracy and public education, does not imply a lousy allocation of resources. On the contrary, the efficient allocation is characterized precisely by this pattern.

The paper also stresses the key role that the quality of institutions plays in understanding the weak impact of human capital in growth regressions, disclosed empirically by Rogers

[3]We refer to skilled workers as those who have reached skills through an education process. We define human capital as the number of skilled workers. Thus, in our model, human capital and skilled labor are synonyms.

[4]This result is consistent with the stylized fact that governments in developing countries have severe trouble raising public revenues, implying a low effective tax rate. See Easterly and Rebelo (1993.a, 1993.b), Gordon and Li (2009), Besley and Persson (2014), and Besley et al. (2021).

[5]This feature is in accordance with the stylized fact that a significant portion of skilled workers in developing countries is engaged in the public sector. See Gelb, Knight, and Sabot (1991), Pritchett (2001), Banerjee (2006), Schmitt (2010), and Schündeln and Playforth (2014).

(2008).To do that, we study the effect of an institutional improvement, measured as a rise in the productivity of the tax collection technology. A possible interpretation of this change is that bureaucrats use their time more efficiently because they spend less time in unproductive or rent-seeking activities. In this sense, we do not exclude the possibility of unproductive uses of human capital in the public sector that previous contributions have stressed (see Ehrlich and Lui, 1999; Mauro, 2004; Blackburn, Bose, and Haque, 2006; Barseghyan and Guerdjikova, 2011). Another possible interpretation is that economic institutions, such as firms, become more transparent, making it easier for bureaucrats to monitor and implement tax duties. In any case, when tax collection technology becomes more productive, the government requires fewer bureaucrats, which generates a reallocation of skilled workers into the production sector and, consequently, a drop in the skill premium and the return on human capital. Thus, human capital decreases in the long run while production and GDP rise. Hence, at the steady-state (long-run equilibrium), countries with better institutional quality will have less human capital but more GDP per capita. As a result, if several countries are at the steady-state and have different institutional quality levels, a misleading negative correlation would arise between the human capital per capita and the GDP per capita.

Summarizing, our paper contributes to understanding the controversial empirical relationship between human capital and economic growth from two different perspectives: first, from a dynamic perspective, when the diversion of human capital to the public sector in the first stage of development involves a low impact of human capital on production and GDP and; second, from a country-wise comparative, when differences in institutions among countries may generate a misleading negative correlation between GDP and human capital.

The theory presented in the paper is consistent with different empirical facts presented in section 11. First, the model predicts that the share of skilled workers (bureaucrats and teachers) increases during the first stage of development up to the point in which the effective tax rate coincides with the statutory tax rate. Afterwards, the government does not require more bureaucrats, and the share of skilled workers hired decreases. Consistent with it, for a large sample of countries, we document a hump-shaped relationship between the percentage of total skilled workers allocated to the public sector (bureaucracy and public education) and the share of skilled workers in the economy. Second, the model predicts that institutional improvements reduce the per capita human capital allocated to the public sector. For the same sample of countries, we test this hypothesis using several measures of institutional quality. In all cases we find that institutional quality is negatively related to the share of total skilled workers allocated to the public sector and that the hump-shaped relationship is robust to the introduction of this variable.

Our paper provides a theory of how the fiscal system and public education evolve along development and how these processes interact with human capital formation. This theory is in line with the mainstream literature of Political Science, which emphasizes both the importance of a competent bureaucracy for economic development and the link between the development of bureaucracy and human capital (see the classical contribution by Gerth and Mills, 1970; Max Weber, 1978; and the empirical evidence by Hollyer, 2011). Furthermore, our paper is also in line with the literature that evidences that the quality of bureaucrats is vital for achieving a successful development process (see Wade, 1990; World Bank, 1993; Evans, 1995; Rauch and Evans, 2000).

Galor and Moav (2006) explain how the complementarity between physical and human capital in production incentives capitalists to support the development of the public education system, while Galor et al. (2009) point out the role of landowners in retarding the emergence of a public education system. These contributions provide theoretical grounding and empirical evidence about the rise in tax collection in association with human capital formation. Our paper analyzes this relationship from a totally different perspective: the feedback process generated by the complementarities between the development of the fiscal system, the public education system, and the formation of human capital.

Our paper is also related to the literature that emphasizes the importance of upper-tail human capital. Some authors consider the role of human capital (literacy) during the Industrial Revolution as minor (Sandberg, 1979; Mitch, 1993; Galor, 2005). However, these studies use education or literacy as a skill measure of the average worker. Recent studies claim that those measures may hide the role of engineers and talented entrepreneurs at the top of the skill distribution (Hanushek and Kimko, 2000; Mokyr, 2005; Mokyr and Voth, 2009; Gennaioli et al., 2013). In this respect, Squicciarini and Voigtländer (2015) find that initial literacy levels, though associated with development in the cross-section, do not predict growth. In contrast, they find that upper-tail knowledge raises productivity in innovative industrial technology. More recently, Goñi (2022) found similar results when analyzing the role of teachers during the industrial revolution in England.

Other papers that investigate the relevance of the allocation of human capital to understand growth include Ehrlich, Li, and Liu (2017), that emphasize the role of innovative entrepreneurial as an engine of growth, and Ehrlich, Cook, and Yin (2018), that emphasize the importance of the quality of higher education. These papers offer new channels to test the relationship between human capital and economic growth from an empirical point of view. The second contribution is specially related to our work since it stresses the role of institutional changes, the quality of the education system, and the importance of public education in generating growth through human capital accumulation. Our paper adopts these ideas but focuses on the way of financing public education and the feedback relationship between public education and the fiscal system.

The organization of the rest of this paper is as follows. Section 2 presents a model where human capital is used to produce: goods, education (teachers), and public revenues (bureaucrats). Section 3 lays out the behavior and decisions of agents in the economy. Section 4 shows the resulting allocation of human capital among the three sectors. Section 5 analyzes the impact of human capital on GDP per capita. Section 6 defines equilibrium. Section 7 examines the dynamic behavior of the economy. Section 8 explores the effect of an institutional change. Section 9 analyzes the optimal allocation of the economy. Section 10 incorporates a public sector wage premium in the model. Section 11 discusses different pieces of empirical evidence related to the model. Finally, section 12 concludes. The appendix includes all the proofs.

2. A three-sector model

Time is continuous and endless and indexed by $t \in \Re_{+}$. There are two production factors[6]: human capital (or skilled labor) and raw (unskilled) labor. There is a continuum of workers, which are either skilled (workers with human capital) or unskilled, having each type of worker one unit of her labor type. We refer to skilled workers as those who have reached skills through the education process (which we will explain later on). We define human capital as the number of skilled workers so that we can use both terms indistinctly. The per capita amount of human capital (or per capita number of skilled workers) is denoted by h, which implies that the per capita number of unskilled workers is $1 - h$.

There are three sectors in the economy:

- Production of consumption goods: uses human capital and unskilled labor. The per capita amount of human capital and unskilled labor devoted to production are denoted respectively by h_y and l.
- Production of human capital (education system): agents are born unskilled; if they want to become skilled workers, they must engage in an education process. The production of human capital requires human capital and unskilled labor. The skilled

[6] We want to focus on the dynamics of the human capital throughout the transition to the steady state equilibrium and, especially, on the human capital reallocation among the different sectors of the economy. For this reason, we simplify the model by assuming that the unique reproducible factor is the human capital. This simplification assumption is very reasonable since introducing another reproducible factor would not alter the human capital reallocation mechanisms along the transition.

workers involved in the education system will be called teachers. The per capita number of skilled workers devoted to producing human capital (teachers) is denoted h_h. The education system also requires unskilled workers who become skilled workers in the future. Such unskilled workers are called students. The per capita number of students is denoted by s. The government provides education. The education system is public, financed by taxes, and the government recruits teachers.

- Production of tax revenues (tax collection): collecting taxes is costly but necessary since taxes finance the public education system. Tax collection requires human capital. Skilled workers recruited by the government to collect taxes are called bureaucrats, and the per capita number of them is denoted by h_b.

To sum up, there are two factors: human capital and unskilled labor. Human capital may be used to produce goods, h_y, to produce human capital (teachers), h_h, or to collect taxes (bureaucrats), h_b. While unskilled labor may be used to produce goods, l, or human capital (students), s.

2.1. Households

There are many identical households, each one of them with a continuum of agents of measure $N(t)$. Fertility rate is constant and denoted by $b > 0$. Agents survive next period with probability $1 - m$, being $m \in (0, 1)$ the mortality rate. This implies that population grows at a constant rate $n \equiv b - m \geq 0$. Thus, $N(t)$ evolves according to the birth and the mortality rate:

$$\dot{N}(t) = bN(t) - mN(t) = (b - m)N(t) = nN(t)$$

Households are composed by skilled workers $H(t)$, unskilled workers $L(t)$ and students $S(t)$:

$$N(t) = H(t) + L(t) + S(t)$$

In per capita terms:

$$h(t) + l(t) + s(t) = 1$$

Household's preferences are given by a time separable utility function:

$$\int_0^\infty N(t)u\left(c(t)\right)e^{-\rho t} = N_0 \int_0^\infty u\left(c(t)\right)e^{-(\rho-n)t}$$

where $c(t)$ denotes the household's consumption per capita at period t, $\rho > n$ denotes the utility discount rate and the utility function $u\left(.\right)$ is the CES utility function:

$$u(c) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \text{if } \sigma \in (0,1) \cup (1, +\infty) \\ \ln c & \text{if } \sigma = 1 \end{cases}$$

2.2. Production of consumption goods:

The production technology of the consumption good is given by the Cobb-Douglas production function:

$$y(t) = A(t)l(t)^{1-\alpha}h_y(t)^{\alpha} \tag{1}$$

where $y(t)$ denotes production of goods per capita, $h_y(t)$ the human capital per capita devoted to production of goods, $l(t)$ the unskilled labor per capita devoted to production of goods at t and, $\alpha \in (0,1)$ is the human capital share. Finally, $A(t) \in R_{++}$ is the total factor productivity, which increases with the per capita number of skilled workers engaged in production: $A(t) = B\widetilde{h}_y(t)^{\beta}$, where $B > 0$, $\beta \in (0, 1-\alpha)$ and $\widetilde{h}_y(t)^{\beta}$ represents the external effect of human capital on the total factor productivity. Thus, the contribution of human capital to the production exceeds the private return of the factor, generating positive spillovers over the productivity of the economy.

2.3. Production of human capital (education system):

Agents are born unskilled; if they want to become skilled workers, they must engage in a costly education process. Individuals have to devote their whole time to education during one period. Agents receiving an education are called students, and the per capita number of students is denoted by s. Furthermore, a student reaches human capital and becomes a skilled worker with probability $\mu(h_h/s)$, which depends on the ratio of teacher/student, h_h/s. Teachers are provided by the government. Thus, the number of skilled workers behaves according to the following law of motion:

$$\dot{H}(t) = \mu\left(\frac{h_h(t)}{s(t)}\right) S(t) - mH(t)$$

where $S(t)$ is the total number of students. The above equation means that the total number of skilled workers, $H(t)$, increases with the number of unskilled workers that acquire human capital through the education process, $\mu\left(h_h(t)/s(t)\right) S(t)$ and decreases with the number of skilled workers that die, $mH(t)$. If we rewrite the above equation in per capita terms we get:

$$\dot{h}(t) = \mu\left(\frac{h_h(t)}{s(t)}\right) s(t) - bh(t) \tag{2}$$

The probability that a student becomes a skilled worker is as follows:

$$\mu\left(\frac{h_h(t)}{s(t)}\right) = \begin{cases} \left(\frac{h_h(t)}{s(t)}\right)^{\xi} & \text{if } \left(\frac{h_h(t)}{s(t)}\right) \leq 1 \\ 1 & \text{if } \left(\frac{h_h(t)}{s(t)}\right) > 1 \end{cases}$$

Note that the probability of effectively reaching the skills, $\mu(.)$, is decreasing in the parameter $\xi \in (0,1)$, and that when ξ tends to zero, then such probability becomes one, $\lim_{\xi \to 0} \mu(h_h/s) = 1$. Thus, parameter ξ can be consider as an inverse index of the quality of the educational system: the lower ξ, the better the performance of the educational system is.

2.4. Production of public revenues (tax collection):

The government hires a certain number of skilled workers as teachers to produce human capital and provides income transfers to households, tr. To finance these expenditures, the government fixes a "statutory" tax rate, $\overline{\tau}$, on the earnings running from the human capital activities[7]. However, the government needs to hire bureaucrats to collect taxes. Individuals will not pay taxes if there is no bureaucracy to manage and control the tax collection. Thus, the effective tax rate that individuals pay depends positively on the number of bureaucrats the government hires. There is a technology that translates bureaucratic efforts into effective public revenues. . In particular, the effective tax rate paid and, so it produces public revenues in period t, is as follows:

$$\begin{aligned} \tau\left(h_b(t)\right) &= \begin{cases} \Gamma\left(h_b(t)\right)^{\gamma} & \text{if } h_b(t) < \overline{h}_b \\ \overline{\tau} & \text{if } h_b(t) \geq \overline{h}_b \end{cases} \\ \Leftrightarrow \tau\left(h_b(t)\right) &= \min\left\{\Gamma\left(h_b(t)\right)^{\gamma}, \overline{\tau}\right\} \end{aligned} \tag{3}$$

where $\overline{h}_b \equiv \left(\frac{\overline{\tau}}{\Gamma}\right)^{\frac{1}{\gamma}}$, $\tau\left(h_b(t)\right)$ denotes the effective tax rate, which is paid by individuals at period t, $h_b(t)$ is the amount of human capital per capita devoted to the bureaucracy (per capita number of bureaucrats), $\Gamma > 0$ and $\gamma \in (0,1)$. It is assumed that the higher the number of the bureaucrats assigned to manage the tax collection is, the higher the effective tax rate is and so, the higher the amount of public revenues. There is a maximum number of bureaucrats, $\overline{h}_b$, that makes the effective tax rate, $\tau\left(h_b(t)\right)$, equal to the statutory tax rate, $\overline{\tau}$.

[7] In this version of the model we assume that fiscal policy rules are fixed along time. Later on, in section 9, we will analyze the optimal fiscal policy, in which fiscal policy rules are endogenous; that is, we will characterize the policies developed by a benevolent social planer that maximizes social welfare (the utility of households). In such a section, we will show that optimal fiscal policy rules resemble the ones defined in the benchmark model.

3. Agents' decisions

3.1. Households

The households' optimization problem is as follows:

$$\max_{\{c(t),s(t)\}_{t=0}^{\infty}} \int_0^{\infty} u\left(c(t)\right) e^{-(\rho-n)t} dt \tag{4}$$

$$c(t) = w_h(t)\left(1-\tau(t)\right) h(t) + w(t)\left(1-h(t)-s(t)\right) + tr(t) \tag{5}$$

$$\dot{h}\left(t\right) = \mu\left(t\right) s\left(t\right) - bh\left(t\right) \tag{6}$$

$$h\left(0\right) > 0$$

where $w(t)$ and $w_h(t)$ denote respectively the wage of the unskilled labor and the wage of skilled workers at period t. Thus, households maximize their utility subject to two constraints: (i) the budget constraint (equation 5), that is, the expenditure in consumption $c(t)$ should be equal to their disposable income that comes from human capital income, $w_h(t)h(t)$, and unskilled labor income, $w(t)\left(1-h(t)-s(t)\right)$, minus taxes $\tau(t)w_h(t)h\left(t\right)$ plus transfers provided by the government $tr(t)$ and; (ii) the accumulation equation of human capital (equation 6).

The Euler Equation and the transversality condition associated to the households' optimization problem are:

$$\frac{\dot{c}\left(t\right)}{c\left(t\right)} = \frac{1}{\sigma}\left[\frac{w_h\left(t\right)\left(1-\tau(t)\right)-w\left(t\right)}{p_h(t)} + \frac{\dot{p}_h(t)}{p_h(t)} - m - \rho\right] \tag{7}$$

$$\lim_{t\to+\infty} \frac{1}{c(t)} e^{(\rho-n)t} p_h(t)h(t) = 0 \tag{8}$$

where $p_h(t) = \frac{w(t)}{\mu(t)}$ is the marginal cost of producing one unit of human capital. Such cost is equal to the amount of unskilled labor required to produce one unit of human capital, $1/\mu\left(t\right)$, multiplied by the price of use of the unskilled labor (the opportunity cost), $w(t)$. The first of the above conditions is the Euler equation. The speed at which consumption grows depends positively on the return of investment in human capital, $\frac{w_h(t)(1-\tau(t))-w(t)}{p_h(t)} + \frac{\dot{p}_h(t)}{p_h(t)}$, and negatively on the discount rate of the household's utility, ρ, and the "depreciation rate" of the human capital, that is, the mortality rate, m. Notice that the return of the human capital takes the form of the return of an asset: the first part, $\frac{w_h(t)(1-\tau(t))-w(t)}{p_h(t)}$, captures the direct return of investment in human capital and the second part, $\frac{\dot{p}_h(t)}{p_h(t)}$, measures the possible "capital gains" derived from changes in the price of human capital. Note that individuals care about the ex ante return to human capital, $\mu(t)\frac{w_h(t)(1-\tau(t))-w(t)}{w(t)}$, instead the ex post return, $\frac{w_h(t)(1-\tau(t))-w(t)}{w(t)}$. In other words, individuals are aware that there exists a certain probability of not acquiring human capital, $\mu(t)$, when they invest in it. The second equation is the standard transversality condition.

3.2. Firms (in the production sector):

Firms behave competitively and hire the number of workers and human capital that maximize their profits:

$$\max_{L(t),H_y(t)} A(t)L(t)^{1-\alpha}H_y(t)^{\alpha} - w_h(t)H_y(t) - w(t)L(t) \tag{9}$$

where $L(t)$ and $H_y(t)$ denote respectively the amount of unskilled labor and human capital hired by the firm at period t. The first-order conditions of the above problem are:

$$\begin{aligned} \alpha A(t)\left(\frac{L(t)}{H_y(t)}\right)^{1-\alpha} &= w_h(t) \\ (1-\alpha)\,A(t)\left(\frac{H_y(t)}{L(t)}\right)^{\alpha} &= w(t) \end{aligned}$$

That is, firms hire a factor up the point in which the marginal productivity of such factor is equal to its price. These first-order conditions may be rewritten in per capita terms:

$$\alpha A(t)\left(\frac{l(t)}{h_y(t)}\right)^{1-\alpha} = w_h(t) \tag{10}$$

$$(1-\alpha)\,A(t)\left(\frac{h_y(t)}{l(t)}\right)^{\alpha} = w(t) \tag{11}$$

3.3. Government

Human capital is assumed to be perfectly substitutable among sectors and there is perfect competition. Thus, wages of all skilled workers are identical, independently of the sector in which they work (production sector, bureaucracy or public education)[8]. The government

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[8] Some empirical papers, such as Depalo, Giordano, and Papapetrou (2015), has shown that wages in the public sector tends to be higher than in the private sector. We extend the model in section 10 to introduce a wage premium in the public sector. However, such extension does not modify at all the results of the model.

budget constraint is as follows:

$$\tau\left(h_b\right) w_h\left(h_y+h_h+h_b\right)=w_h\left(h_h+h_b\right)+tr \tag{12}$$

The left-hand of the equation is the total public revenues of the government in per capita terms that come from the taxation over the human capital income. Per capita public revenues are defined by the effective tax rate multiplied by the per capita number of skilled workers' income. The right-hand of the equation represents the government expenditures: (*i*) per capita expenditure in public education, $w_h h_h$, that is, the wages paid to teachers; (*ii*) wages per capita paid to bureaucrats, $w_b h_b$ and; (*iii*) per capita amount of transfers devoted to households, tr, which represents all the government expenditures that are not devoted either to paying bureaucrats or teachers[9]. For simplicity, we assume that the government devotes a fraction, $\lambda \in (0,1)$, of the public revenues to hiring teachers. The remaining tax revenues are dedicated to paying bureaucrats and transfers to households:

$$\begin{aligned}
\lambda\tau\left(h_b\right) w_h h &= w_h h_h &(13)\\
(1-\lambda)\,\tau\left(h_b\right) w_h h &= w_h h_b + tr &(14)
\end{aligned}$$

The objective of the government[10] is to maximize net public revenues, that is, public revenues minus bureaucratic costs incurred to collect those revenues. Thus, the government hires the number of bureaucrats that maximizes the net tax collection:

$$\max_{h_b} T(t) - w_h(t)h_b(t) \tag{15}$$

where $T(t)$ denotes the amount of public revenues (tax collection):

$$T(t)=\tau\left(h_b(t)\right) w_h(t)h(t)=\min\left\{\Gamma\left(h_b(t)\right)^{\gamma},\overline{\tau}\right\} w_h(t)h(t) \tag{16}$$

The solution of the problem is the optimal number of bureaucrats:

$$h_b(h(t))=\begin{cases} (\gamma\Gamma)^{\frac{1}{1-\gamma}}\, h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t)<\overline{h} \\ \overline{h}_b & \text{if } h(t)\geq\overline{h} \end{cases}; \tag{17}$$

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[9]The introduction of the transfers in the model captures the fact that, empirically, payments to the bureaucracy and the education system do not represent all government expenditures. Moreover, it will allow us to derive a simple linear fiscal rule to characterize the relationship between the spending in the education system and public revenues.

[10]We will analyze later on, in section 9, the behavior of the government when its objective is to maximize social welfare (the utility of households).

where $\overline{h} = \frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}$ and $\overline{h}_b = \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}$ denote the threshold levels of respectively human capital per capita and bureaucrats per capita that makes the effective tax rate, $\tau(h_b)$, to coincide with the statutory tax rate. Once the optimal number of bureaucrats is obtained, it is easy to calculate the effective tax rate, per tax revenue per capita and the per capita number of teachers:

$$\tau\left(h_b(h(t))\right) = \begin{cases} \left(\gamma^\gamma\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{\tau} & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{18}$$

$$T(h(t)) = \tau\left(h(t)\right) w_h(t)h(t) = \begin{cases} w_h(t)\left(\gamma^\gamma\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{\tau} w_h(t)h(t) & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{19}$$

$$h_h(h(t)) = \frac{\lambda T((h(t))}{w_h(t)} = \begin{cases} \lambda\left(\gamma^\gamma\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \lambda\overline{\tau}h(t) & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{20}$$

Finally, the transfer payments would be as follows:

$$tr(h(t), w_h(t)) = \begin{cases} \left(1-\gamma-\lambda\right) w_h(t)\left(\gamma^\gamma\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ w_h(t)\left[\left(1-\lambda\right)\overline{\tau}h(t) - \overline{h}_b\right] & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{21}$$

Notice that when $h(t) < \overline{h}$, the share of the tax collection devoted to paying bureaucrats is equal to γ, while the share devoted to paying teachers is λ. Thus, in order to guarantee the existence of non negative transfer payments we assume that the fraction of taxes devoted to bureaucrats, γ, plus the fraction devoted to teachers, λ, are together equal or smaller than one: $\gamma + \lambda \leq 1$.

4. The allocation of human capital among sectors

Once we determine the optimal number of bureaucrats (equation 17), h_b, and the number of teachers (equation 20), we obtain the amount of human capital that is devoted to production of goods, h_y, as the remaining amount of human capital:

$$h_y(h(t)) = h(t)\text{-}h_b(h(t))\text{-}h_h(h(t)) = \begin{cases} h(t)\text{-}(\gamma+\lambda)\left(\gamma^\gamma\Gamma\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ (1\text{-}\overline{\tau}\lambda)\, h(t)\text{-}\overline{h}_b & \text{if } h(t) \geq \overline{h} \end{cases} \tag{22}$$

We may also define the allocation of human capital in its three possible uses: bureaucracy (equation 17), education (equation 20) and production (equation 22), as ratios with respect to the total amount of human capital:

$$\frac{h_b(h(t))}{h(t)} = \begin{cases} (\gamma\Gamma)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \frac{\overline{h}_b}{h(t)} & \text{if } h(t) \geq \overline{h} \end{cases} \tag{23}$$

$$\frac{h_h(h(t))}{h(t)} = \begin{cases} \lambda\,(\gamma^\gamma\Gamma)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \overline{\tau}\lambda & \text{if } h(t) \geq \overline{h} \end{cases} \tag{24}$$

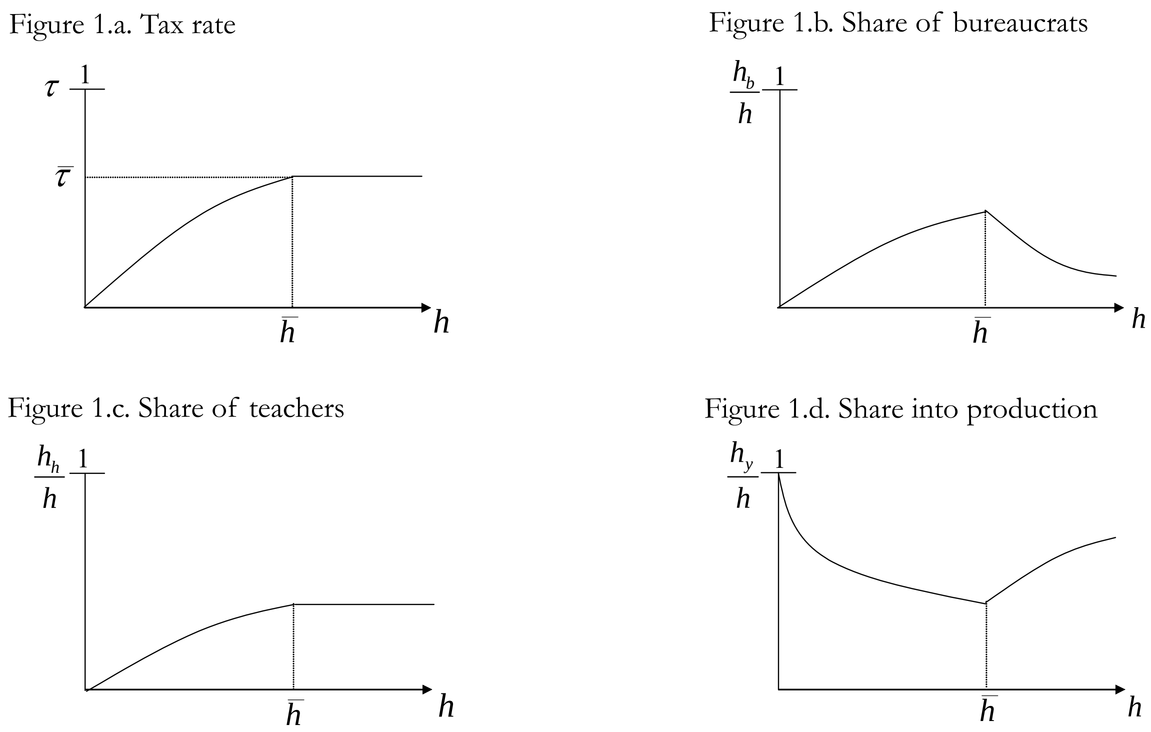
$$\frac{h_y(h(t))}{h(t)} = \begin{cases} 1-(\gamma+\lambda)\,(\gamma^\gamma\Gamma)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ (1-\overline{\tau}\lambda) - \frac{\overline{h}_b}{h(t)} & \text{if } h(t) \geq \overline{h} \end{cases} \tag{25}$$

Figure 1 displays these ratios. The evolution of the three different uses of human capital depends on the evolution of the effective tax rate. As figure 1.a shows, the effective tax rate is an increasing function of the human capital per capita until reaching the threshold level, $\overline{h}$, in which the effective tax rate coincides with the statutory tax rate $\overline{\tau}$ (see equation 18). Beyond this threshold, the effective tax rate is constant and equal to the statutory tax rate. When the level of human capital per capita is low (in poor countries), collecting taxes is expensive because it requires human capital, which is scarce.

Consequently, the effective tax rate is low and human capital is mostly devoted to production (see figure 1.a and figure 1.d). However, insofar human capital rises and becomes less scarce, the government hires more bureaucrats, implementing a higher effective tax rate. Thus, the per capita number of bureaucrats and the effective tax rate rise with human capital (see figures 1.a and 1.b), which, in turn, allows to hire of an increasing number of teachers (see figure 1.c). Therefore, the increasing effective tax rate generates a reallocation of human capital from the private to the public sector. Wherefore, the share of human capital devoted to production decreases with human capital (see 1.d), whereas the shares of both bureaucrats and teachers over the human capital rise with it. It may even happen that the amount of human capital devoted to production decreases with human capital not only as a share of human capital but also in per capita terms (this would be the case if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$, see equation 22). These dynamics happen until the human capital reaches the threshold level $\overline{h}$, in which the effective tax rate coincides with the statutory tax rate.

Once the effective tax rate reaches its statutory level, the tax rate remains fixed independently of the human capital per capita in the economy (see figure 1.a). Thus, the government only hires the per capita number of bureaucrats needed to collect the statutory tax rate. Consequently, the share of bureaucrats over human capital decreases when human capital per capita rises, as figure 1.b displays. Since a constant fraction λ of tax revenues are devoted to hiring teachers, and the tax rate is fixed at the statutory level, the share of teachers over human capital remains constant (see figure 1.c). Since the share of teachers

Fig. 1.— Allocation of human capital



in human capital remains steady but the share of bureaucrats declines with human capital, the share of human capital devoted to the public sector declines with human capital. Consequently, the amount of human capital allocated to the production increases, as figure 1.d displays.

5. The impact of human capital on GDP per capita

The previous section shows that, in earlier stages of development (until reaching the threshold of human capital per capita $\overline{h}$), an increasing amount of human capital is allocated to the public sector (bureaucrats and teachers). This reallocation leads to a lower portion of skilled workers in the private sector and, consequently, to a weak effect of human capital in production. Furthermore, it follows from the analysis that if the government's demand for skilled workers is high enough, the human capital per capita devoted to production, h_y, may even decrease with human capital per capita (if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$), and consequently, production may decrease with human capital as well. In summary, our results indicate that the effect of human capital on production at the beginning of the development process may be weak and even negative. However, production in this model is different from GDP because GDP includes skilled workers' incomes in the public sector (both teachers and bureaucrats). In this section, we will study the impact of human capital on GDP per capita.

Definition 1 *GDP per capita is defined as follows:*

$$gdp \equiv wl + w_h h = \underbrace{y}_{\text{consumption (production of goods)}} + \underbrace{w_h \left(h_h + h_b\right)}_{\text{Public Expenditure on public workers}}$$

Note that GDP include both production of goods (consumption) and public expenditure, which includes the expenditure in the education system (the wages of teachers) and the expenditure in the tax collection system (the wage of bureaucrats).

Proposition 2 *The GDP per capita may be written as a function of the skilled labor per capita, h, and unskilled labor, l, and the ratio human capital devoted to production-human capital, h_y/h, that is, $gdp\left(h, l, \frac{h_y}{h}\right)$. Moreover, there exists $\overline{\beta} < (1-\alpha)$, defined in the appendix, such that if $\beta \geq \overline{\beta}$ then the GDP per capita is an increasing function of: the ratio human capital devoted to production-human capital, h_y/h, the skilled labor per capita, h, and the amount of unskilled labor, l, that is, $\frac{\partial gdp\left(h,l,\frac{h_y}{h}\right)}{\partial\left(\frac{h_y}{h}\right)} > 0$, $\frac{\partial gdp\left(h,l,\frac{h_y}{h}\right)}{\partial h} > 0$ and, $\frac{\partial gdp\left(h,l,\frac{h_y}{h}\right)}{\partial l} > 0$.*

Proposition 3 *The GDP per capita may be written as a function of the skilled labor per capita, h, and per capita number of students, s, that is, $gdp\,(h, s)$, where $\frac{\partial gdp(h,s)}{\partial s} < 0$. Moreover, in a subset of the parameter space there exists $\varepsilon > 0$ such that if $h \in \left(\overline{h} - \varepsilon, \overline{h}\right)$ then $\frac{\partial gdp(h,s)}{\partial h} < 0$.*

According to proposition 2, GDP per capita increases not only with the amount of per capita factors, both skilled and unskilled labor(h and l respectively) but also with the portion of skilled labor devoted to production of goods, h_y/h. This is a relevant finding because the significant absorption of skilled workers by the public sector in the earlier stages of development reduces the ratio h_y/h and the growth of GDP per capita. This result implies that the impact of human capital on GDP growth may be weak in the earlier stages of development due to the diversion of human capital from production to the public sector. Furthermore, proposition 3 establishes that the GDP per capita may decrease with human capital (in a certain interval). Thus, the effect of human capital on GDP per capita may be not only weak but even negative in earlier stages of development due to the increasing absorption of human capital by the public sector.

6. The definition of equilibrium

The equilibrium definition is standard: equilibrium occurs when agents maximize their objective functions and markets clear. Since all households and firms are alike, we may define equilibrium in per capita terms.

Definition 4 *Given the initial condition h_0, a competitive equilibrium is an allocation $\{c(t), s(t), h(t), h_y(t), l(t), A(t), h_b(t), h_h(t), tr(t)\}_{t=0}^{\infty}$ and a vector of prices $\{w(t), w_h(t)\}_{t=0}^{\infty}$ such that $\forall t$:*

- Households maximize their utility, that is, $\{c(t), s(t), h(t)\}_{t=0}^{\infty}$ is the solution of the household's maximization problem (4).
- Firms maximize profits, that is, $h_y(t), l(t)$ is the solution of the firm's maximization problem (9).
- The government chooses the amount of human capital devoted to bureaucracy, $h_b(t)$, which maximizes the net public revenues (15) and chooses the amount of human capital devoted to the public education system (teachers), $h_h(t)$, and the transfer payments according to fiscal policies rules (13) and (14).
- Human capital market clears: $h(t) = h_y(t) + h_h(t) + h_b(t)$.
- Unskilled labor market clears: $1 - h(t) - s(t) = l(t)$.
- Goods market clears: $y(t) = A(t)h_y(t)^{\alpha}l(t)^{1-\alpha} = c(t)$
- Total factor productivity is: $A(t) = Bh_y(t)^{\beta}$

Definition 5 *steady-state equilibrium is an equilibrium in which both the allocation and the vector of prices always remain constant over time.*

7. Dynamic behavior

The dynamic behavior of this economy could be characterized by the dynamics of the consumption and the human capital. We now proceed to define the dynamic system of the economy.

7.1. Dynamic system

The dynamic system of this economy consists of the human capital accumulation equation (6), the Euler equation (7) and the transversality condition (8):

$$\begin{aligned}
\dot{h}(t) &= \mu\left(\frac{h_h(h(t))}{s(t)}\right)s(t) - bh(t) && (26)\\
\frac{\dot{c}(h(t),s(t))}{c(h(t),s(t))} &= \frac{1}{\sigma}\left[\frac{w_h(h(t),s(t))\,(1\text{-}\tau(h_b(h(t))))\text{-}w(h(t),s(t))}{p_h(h(t),s(t))} + \frac{\dot{p}_h(h(t),s(t))}{p_h(h(t),s(t))} - m - \rho\right]\\
&\lim_{t\to+\infty}\frac{1}{(c(h(t),s(t)))}e^{-\rho t}p_h(h(t),s(t))\,h(t) = 0
\end{aligned}$$

where $w_h(h,s)$ is the marginal product of human capital in the production sector, which coincides at equilibrium with its wage; $w(h,s)$ is the marginal product of unskilled labor in the production sector, which coincides at equilibrium with its wage; $p_h(h,s)$ is the marginal cost of the production of one unit of human capital; and $c(h,s)$ is the household's consumption per capita after tax/transfers income, this is,

$$w(h,s) = (1-\alpha)A(h_y(h))\left(\frac{h_y(h)}{1-h-s}\right)^{\alpha}; \quad w_h(h,s) = \alpha A(h_y(h))\left(\frac{1-h-s}{h_y(h)}\right)^{1-\alpha}; \quad p_h(h,s) = \frac{w(h,s)}{\mu\left(\frac{h_h(h)}{s}\right)}$$
$$c(h,s) = w_h(h,s)\left[1-\tau(h_b(h))\right]h + w(h,s)(1-h-s) + tr(w_h(h,s),h)$$
$$\text{and } A(h_y(h)) = Bh_y(h)^{\beta}$$

The above dynamic system may be rewritten in terms of $h(t)$ and $s(t)$:

$$\begin{aligned}
\dot{h}(t) &= F_h(h(t),s(t)) && (27)\\
\dot{s}(t) &= F_s(h(t),s(t)) && (28)
\end{aligned}$$

where $F_h(.)$ is the accumulation equation of human capital (26) and $F_s(.)$ is defined in the appendix.

Proposition 6 *If $\xi + \gamma \leq 1$, there exists $\overline{\Gamma} > 0$ such that if $\Gamma > \overline{\Gamma}$ then there is a unique steady-state equilibrium and $\tau^{ss} = \overline{\tau}$ ($h^{ss} > \overline{h}$).*

Note that in this model, there exists a feedback process: more human capital involves more bureaucrats, which, in turn, collect more taxes, allowing the government to hire more teachers, which, in turn, increase the return of investing in education, promoting human capital accumulation, and so on. This feedback process might generate a virtuous circle but also a vicious circle that ends in a poverty trap: a low level of human capital implies few bureaucrats that collect few taxes, which involves a reduced number of teachers that, in turn, reduces the return on human capital, discouraging human capital formation. The two key elements in this feedback process are the public revenues and the education system. Proposition 6 considers these two key elements. If the productivity of the education system is high enough (ξ is low enough) and the productivity of the tax collection technology is high enough (Γ is high enough and γ is low enough), then there are not poverty traps[11]. Instead, a unique steady-state exists in which the effective tax rate is the statutory one. This proposition emphasizes the importance of the quality of the education system and the institutions (measured by the productivity of the tax collection system) for development.

We will concentrate in the case in which there is a unique steady-state, and the effective tax rate coincides with the statutory one at the steady-state. Thus, we assume from now on that $\xi + \gamma \leq 1$ and $\Gamma > \overline{\Gamma}$.

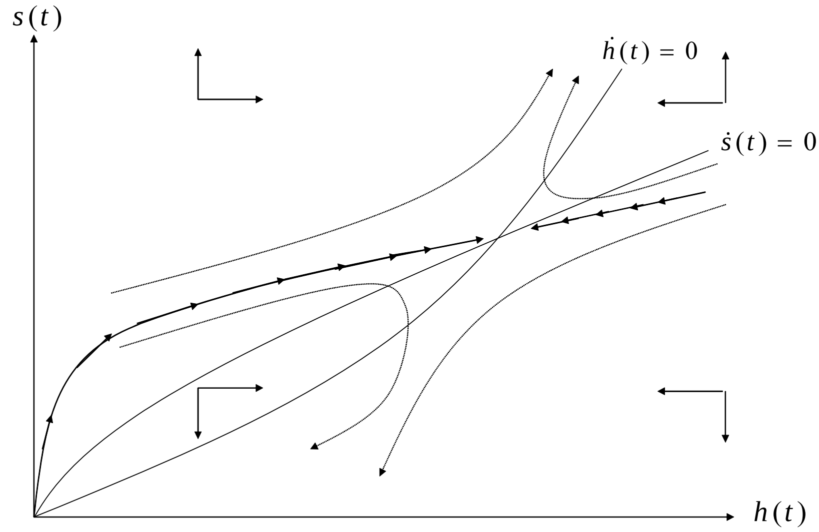
Proposition 7 *There exists $\overline{\sigma}$ such that, if $\sigma > \overline{\sigma}$, then the steady-state equilibrium is a saddle point and $s(t)$ increases when $h(t) < h^{ss}$ and decreases when $h(t) > h^{ss}$.*

The saddle point dynamics imply a unique path that converges to the steady-state. Therefore, given the initial level of human capital per capita, there is only one equilibrium trajectory that converges to the steady-state. The phase diagram in figure 2 shows that the dynamic behavior of the economy is characterized by the typical saddle point dynamic. Wherefore, when the initial amount of human capital per capita is lower than the steady-state level, the number of students grows throughout the equilibrium path, converging to its steady-state level. The opposite happens when the amount of human capital per capita is larger than the steady-state level.

The evolution of the number of students along the transition depends significantly on the elasticity of substitution of the utility function ($1/\sigma$). To see this, consider that the initial human capital per capita, $h(0)$, is below the steady-state level, $h(0) < h^{ss}$. So, due to the relative scarcity of human capital, education returns are high. To determine the relationship between human capital and the number of students, we have to consider two effects: a

[11] If $\xi + \gamma > 1$, the economy may converge to the trivial steady state, in which the amount of human capital is zero. In this case, the low quality of the public education system and the scarcity of teachers due to the low productivity of tax collection system implies a low probability of a student becoming a skilled worker and little incentive to invest in education. Consequently, the economy cannot even replace the skilled workers who "depreciate" (die) each period. Thus, the per capita amount of human capital declines each period, converging to zero.

Fig. 2.— Phase diagram



substitution effect and a wealth effect. Insofar countries accumulate human capital, the return of human capital decreases, reducing, in turn, the incentive to have more students in the economy. Thus, a substitution effect would imply a decrease in students. Simultaneously, when countries own more human capital and can afford higher levels of consumption, they tend to have more students since they would like to enjoy higher consumption levels in the future. Thus, a wealth effect would imply an increase in students. The resulting net effect would depend on the relative size of those two effects. However, the relevant case from the empirical point of view is when the number of students increases during the development process. So, the substitution effect should not be too large for the model to reproduce this stylized fact. Wherefore we will concentrate on the case in which the elasticity of substitution is small enough, $\frac{1}{\sigma} < \frac{1}{\overline{\sigma}}$. That is when the parameter sigma is large enough, $\sigma > \overline{\sigma}$.

7.2. Macroeconomic effects: the dynamics of the allocation of human capital among sectors along the transition

We now analyze the allocation of human capital among different sectors throughout the transition to the steady-state. Let's consider that the starting human capital per capita is below the threshold $\overline{h}$, which, in turn, is lower than the steady-state level (see proposition 6). We may differentiate two different stages of development along the transition: (i) in the first stage of development, when human capital per capita is below the threshold $\overline{h}$, the effective tax rate is below the statutory tax rate, and; (ii) in the second stage of development, when human capital per capita is above the threshold $\overline{h}$, the effective tax rate coincides with the statutory one. We define t^* as the moment in which the effective tax rate reaches the statutory tax rate (and human capital per capita reaches the threshold level $\overline{h}$). Thus, the first stage of development will occur from the initial moment of the economy until t^*, and the second one from t^* henceforth. In the first stage of development, the scarcity of human capital precludes the government from hiring enough bureaucrats to implement the effective tax rate. Consequently, the resulting low public revenues do not allow the government to employ many teachers. Since there are few bureaucrats and teachers, most human capital is devoted to production.

However, along the transition, human capital increases. The increasing abundance of human capital allows the government to hire more bureaucrats, increasing the effective tax rate, the tax collection, and consequently, the number of teachers. Therefore, in the first stage of development, the effective tax rate and the shares of bureaucrats and teachers over human capital are increasing (see equations 18, 23, and 24, and figures 3.a, 3.b and 3.c). At the same time, the share of skilled workers devoted to production is declining (see equation 25 and figure 3.d).

Once the statutory tax rate is reached (at the moment t^*), the second stage of development starts. The tax rate is steady at the statutory level (figure 3.a), and the government does not need to increase the per capita number of bureaucrats since it just needs the threshold level $\overline{h}_b$ to implement the statutory tax rate. Because the per capita number of bureaucrats is constant and the human capital per capita rises along the transition, the share of bureaucrats over human capital declines, as figure 3.b displays. Moreover, given that a constant fraction of public revenues is devoted to public education and the tax rate is steady, the share of teachers over human capital remains constant, as figure 3.c shows. Finally, since the percentage of bureaucrats in human capital falls and the percentage of teachers remains constant, the share of total human capital allocated to the public sector declines. Consequently, the share of human capital in the private sector (production) rises in the second stage of development (after the moment t^*).

Figure 4 displays the evolution of factors used in production along the transition to the steady-state. Regarding skilled labor, as explained in section 4, the human capital per capita devoted to production does not necessarily increase when the economy accumulates human capital. Indeed, if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$, the amount of human capital per capita dedicated to the

Fig. 3.— Evolution of human capital allocation

Figure 3.a. Tax rate

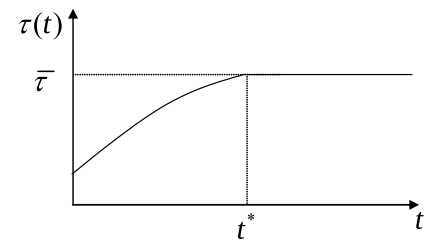


Figure 3.b. Share of bureaucrats

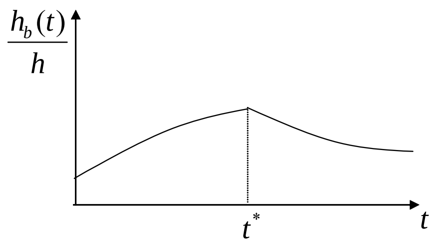


Figure 3.c. Share of teachers

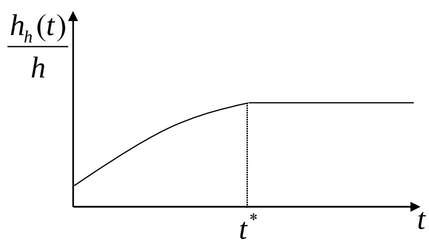
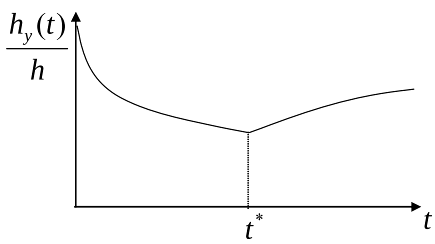


Figure 3.d. Share into production



production behaves not monotonically. Figure 4.a displays such a case. Regarding unskilled labor, note that the per capita amount of human capital and the per capita number of students rise. Consequently, the level of unskilled labor falls ($l = 1 - h - s$), as figure 4.b shows. Similarly, the total per capita number of workers devoted to production (skilled plus unskilled workers) also shows a declining pattern along the transition. To see this result consider the following equation:

$$h_y + l = h_y + 1 - h - s = 1 - h_b - h_h - s = \begin{cases} 1 - (\gamma + \lambda)\left(\gamma^\gamma \Gamma\right)^{\frac{1}{1-\gamma}} h^{\frac{1}{1-\gamma}} - s & \text{if } h < \overline{h} \\ 1 - \overline{h}_b - \lambda \overline{\tau} h - s & \text{if } h \geq \overline{h} \end{cases}; \quad (29)$$

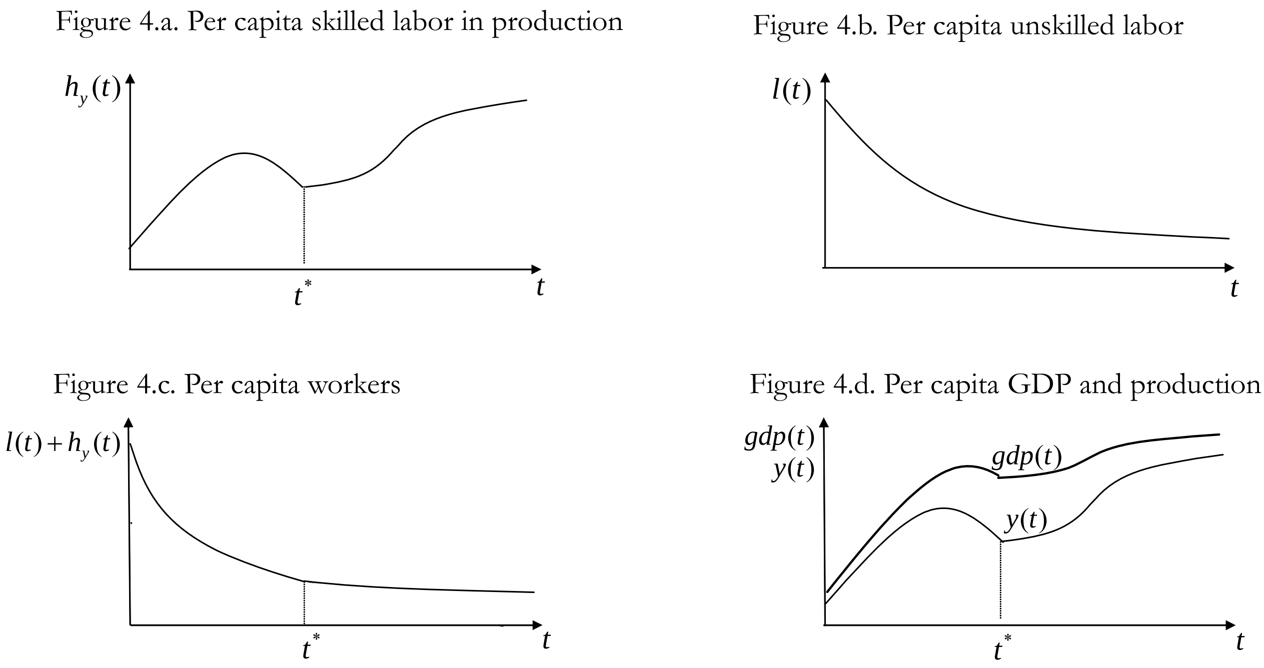
The above equation says that the total (per capita) number of workers devoted to production, both skilled workers h_y and unskilled workers l, is equal to the per capita number of workers minus the workers that do not devote their time to the production: bureaucrats h_b and teachers h_h, in the skilled workers's group, and students s, in the unskilled group, that is, $1 - h_b - h_h - s$. Since the number of bureaucrats, teachers, and students increases in the first stage of development (before t^*), the per capita number of workers devoted to production declines always, as figure 4.c shows (see equation 29).

In the second stage of development (after t^*), the per capita amount of human capital devoted to production increases (see (22) and figure 4.a), and the total number of workers devoted to production decreases at a lower rate than in the first stage of development (see 29 and figure 4.c).

Production per capita may slow down or even not be monotonic along time before t^*. Because at the first stage of development, the per capita number of skilled workers devoted to production is not necessarily monotonic, and the per capita number of unskilled workers and the per capita total number of workers decline along the transition. Figure 4.d displays the case in which production is not monotonic. Thus, the constant reallocation of skilled workers from production to the public sector and the increasing drain of unskilled workers from production to education (the students) generates a slowdown in production. Also, GDP per capita shows a similar evolution as production per capita in the first stage of development. In the light of proposition 2, GDP per capita increases with skilled labor per capita, h, unskilled labor per capita, l, and the ratio of skilled labor used in production/human capital, h_y/h. Because the last two variables decline in the first stage of development, GDP per capita may slow down due to the increasing diversion of human capital from production to public sector activities and the reduction in unskilled labor.

Furthermore, it follows from proposition 3 that, in the first stage of development GDP per capita may even fall along time. Figure 4.d displays such a case. This behavior of GDP per capita may explain the weak effect of human capital on economic performance at the empirical level.

Fig. 4.— Evolution of GDP, production and production factors



In the second stage of development (after t^*), the number of skilled workers in the production sector increases. Considering that they have higher marginal productivity than unskilled workers, the most plausible case is that both production per capita and GDP per capita rises along the transition (figure 4.d), which is consistent with the empirical evidence.

7.3. Microeconomic effects: the evolution of the skill premium along the transition

Empirical studies about the return of human capital at the micro level often use the skill premium as an indicator of the return of human capital, where the skill premium is defined as follows:

$$sp = \frac{w_h - w}{w}$$

Note that in our model, this skill premium definition would be equivalent to the ex-post return of human capital. The reason is simple: households face a risk when investing in human capital. The household's investment in human capital consists of investing in unskilled labor. One unit of unskilled time invested in human capital has an opportunity cost which is the foregone wage of unskilled labor, and a gain (in return) which is the increase in the wage due to the higher wage of skilled labor. However, this gain only exists when the individual successfully obtains human capital (acquiring the skills), implying that the return is ex-post. Thus, in this case, the household invests the wage of an unskilled worker, w, and obtains the wage increase when the worker is skilled, $w_h - w$.

It follows from equations (10) and (11) that:

$$sp = \frac{\alpha}{1-\alpha}\frac{1-h-s}{h_y} - 1 \tag{30}$$

Thus, the skill premium depends on the relative abundance of human capital relative to unskilled labor in the production sector, which is the sector that determines the wages of both the unskilled and the skilled workers. When the initial human capital per capita is below the steady-state level, the unskilled labor used in production constantly declines along the transition since the per capita number of skilled workers and students increases. Thus, if the human capital per capita devoted to production rises along the transition, it would become increasingly abundant relative to unskilled labor. Consequently, the skill premium would decline along the transition. However, as we will show in the next section, the human capital per capita devoted to production not always rises along the transition. The following proposition shows that, even in this case, the skill premium declines along the transition[12].

Proposition 8 *Assume that $h(0) < h^{ss}$, there exists $\widehat{\tau} > \frac{1-\gamma}{\gamma+\lambda}$ such that, if $\overline{\tau} \leq \widehat{\tau}$ then $\forall t \geq 0$ $\dot{sp}(t) < 0$.*

Note that if $\overline{\tau} > \frac{1-\gamma}{\gamma+\lambda}$ the human capital per capita devoted to production does not always rises along the transition (see equation 22). Thus, according to the above proposition, the fact that the human capital devoted to production may decline along the transition does not preclude the decline of skill premium. This result is consistent with the empirical evidence that shows that the return on education declines with the income level per capita (see Psacharopoulos, 1994; Psacharopoulos and Patrinos, 2004; and Strauss and Duncan, 1995).

[12] If we would define the skill premium by considering taxes (the after-tax skill premium), the skill premium would decline even faster along the transition (that is, proposition 8 would hold). The reason is simple: the tax rate increases along the transition and is levied only on skilled labor income. Thus, the rise in tax rate along the transition would contribute to reducing the after-tax skill premium along the transition..

8. Institutional changes: the effect of an improvement in the technology of the bureaucracy

In this section, we evaluate the effect of an institutional improvement by enhancing the government's performance in producing public revenues. More precisely, we analyze the impact of technological progress in the bureaucratic sector through an increase in the parameter Γ. In this context, a technological improvement of the bureaucracy implies that tax collection increases for the same number of bureaucrats, making the effective tax rate closer to the statutory tax rate.

One interesting way of micro-funding the technology in the bureaucratic sector is considering that bureaucrats devote part of their time to unproductive activities. In this case, a reduction in the share of the time dedicated to unproductive activities can be interpreted as an institutional improvement that implies increased tax collection. To analyze this consider the following modification of the tax collection technology (see equation 3):

$$\tau\left(h_b(t)\right) \;=\; \begin{cases} \Upsilon\left[\left(1-\psi\right)h_b(t)\right]^{\gamma} & \text{if } h_b(t) < \overline{h}_b \\ \overline{\tau} & \text{if } h_b(t) \geq \overline{h}_b \end{cases} \qquad \overline{h}_b \equiv \left(\frac{\overline{\tau}}{\Upsilon\left(1-\psi\right)^{\gamma}}\right)^{\frac{1}{\gamma}}$$

where $\Upsilon > 0$ and $\psi \in (0,1)$ is the portion of bureaucratic time devoted to unproductive activities. If we define $\Gamma \equiv \Upsilon\left(1-\psi\right)^{\gamma}$, it is easy to see that the above tax collection technology is the same as the one already presented in equation (3). With this technology, any drop in ψ implies an improvement in the tax collection technology (a rise in Γ). Thus, an equivalent way to interpret an increase in Γ is an institutional improvement that makes bureaucrats devote less time to unproductive activities. There are other alternative ways of interpreting enhancements in technology. For instance, positive institutional changes may also imply considering more transparent institutions or responsible taxpayers that make collecting taxes more manageable and less costly.

An increase in Γ implies that implementing the statutory tax rate requires a smaller amount of human capital (bureaucrats). Thus, human capital reallocates from bureaucracy to production. Furthermore, the increase in the number of skilled workers in the private sector reduces their wages, discouraging human capital accumulation and diminishing the number of students. Consequently, the human capital per capita in the new steady-state is lower.

Nevertheless, despite a lower level of human capital per capita, the human capital per capita used in production is higher. Furthermore, because there are fewer skilled workers and fewer students in the new steady-state, there are also more unskilled workers in the production sector. Therefore, as a result, the increase in the total number of workers in the production sector (both skilled and unskilled) will cause a higher level of production per capita at the new steady-state.

The following proposition establishes that an improvement in institutional quality (represented by an increase in parameter Γ) expands both per capita production and GDP per capita.

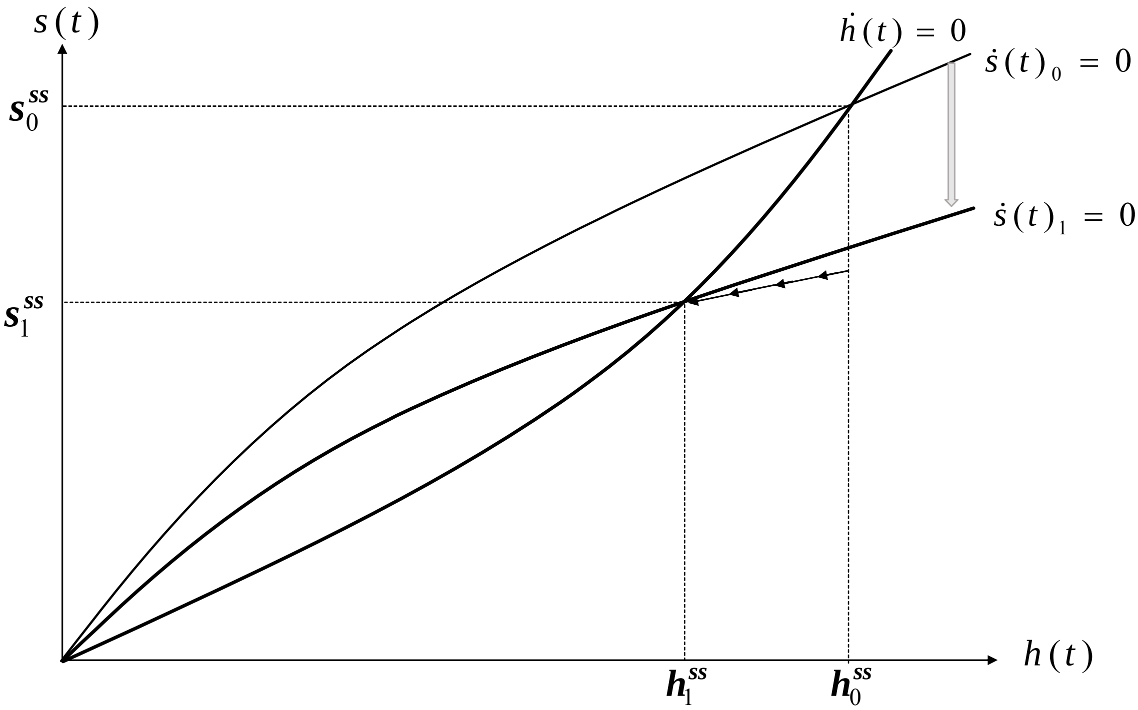
Proposition 9 *If there is a technological improvement in the bureaucratic sector, measured as an increase in* Γ*, then the steady-state levels of both students and human capital decrease and, the amount of human capital devoted to production increases. Moreover, an increase in* Γ *also involves an increase in GDP at the steady-state.*

This proposition shows that there may be a negative relationship between institutional quality and human capital per capita and, at the same time, a positive relationship between institutional quality and GDP per capita. These two relationships together may generate a misleading negative relationship between human capital per capita and GDP per capita. To see this, consider many countries with different degrees of institutional quality, represented by different levels of the parameter Γ and all at the steady-state. Those countries with better institutions (high Γ) have higher levels of production per capita and GDP but lower levels of human capital than those with poorer institutions. Thus, the correlation between GDP per capita and human capital per capita would be negative. However, this result does not mean that human capital does not contribute to production. It simply means that those countries with weaker institutions and consequently with lower production are the ones that require more human capital to produce law enforcement and to encourage tax compliance. Therefore, this negative correlation between GDP per capita and human capital is misleading since the rise in human capital per capita does not involve a fall in GDP per capita. Indeed, the unique engine of growth in this model is human capital. This result sheds some light on the debate about the weak or even negative correlation between human capital and economic performance that has been documented by many empirical papers (see the introduction).

The phase diagram in figure 5 shows the dynamic behavior of the economy after an increase in Γ. An increase in Γ involves a shift to the right of the locus $\dot{s} = 0$. We observe that departing from the initial level of human capital per capita, there exists a unique equilibrium path, which converges to a steady-state with a lower level of human capital and a lower number of students, but with a higher level of production per capita, as proposition 9 establishes.

Figure 6 shows the evolution of human capital devoted to each sector of the economy when Γ increases at period t_0. Because bureaucrats are more efficient in collecting taxes,

Fig. 5.— An institutional improvement



the government can implement the statutory tax rate with fewer bureaucrats. Thus, the government hires fewer bureaucrats (figure 6.a) and spends more resources to provide more transfers to households (see equations 17 and 21). Consequently, skilled workers are reallocated from bureaucracy to the production sector (figure 6.c). This increase in the number of skilled workers in the production sector reduces the wage of skilled workers, discouraging the accumulation of human capital and reducing the number of students and the human capital per capita. The downside of human capital implies a gradual reduction in the per capita number of teachers (figure 6.b), which reduces the return on the human capital, discouraging more human capital accumulation. As a result, the per capita amount of human capital devoted to production declines along the transition. Still, it converges to a higher level than the one at the initial steady-state (as we established in proposition 9 and figure 6.c displays). Finally, reducing students along the transition involves an increase in the amount of unskilled labor per capita devoted to production (figure 6.d). The rise in the amount of both skilled and unskilled labor in production explains the increase in the production per capita and GDP per capita at the new steady-state.

9. Optimal fiscal policy

In previous sections, we analyzed the case of a government that hires bureaucrats to obtain the effective tax rate that maximizes the net tax collection. Public revenues finance all public expenditures: bureaucrats' wages, teachers' wages, and transfer payments to households. For simplicity, we assumed that the government spent a constant fraction of the public revenues on hiring teachers. In this section, we will analyze the case where the government's objective is to maximize social welfare (the utility of households). That is, the government will choose the tax rate that maximizes welfare in this economy. Since statutory tax in this section is not exogenous, we define the (effective) tax rate as follows:

$$\tau = \tau\left(h_b(t)\right) = \Gamma\left(h_b(t)\right)^{\gamma} \tag{31}$$

Fig. 6.— The effect of an institutional improvement

Figure 6.a. Bureaucrats

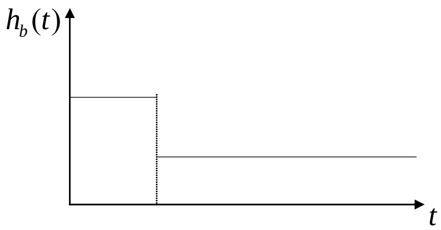


Figure 6.b. Teachers

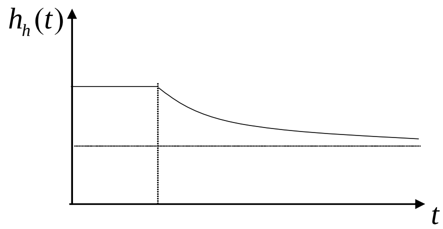


Figure 6.c Human capital in production

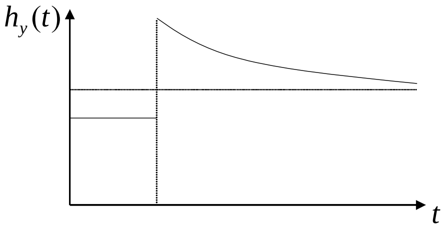
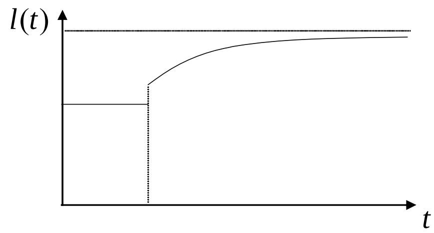


Figure 6.d. Unskilled labor



The budget constraint of the government is similar to the benchmark model but without transfers, which are not included in the present analysis[13]:

$$\tau\left(h_b\right) w_h h = w_h\left(h_h + h_b\right) \tag{32}$$

Using equations (31) and (32) together with the constraint $h = h_y + h_h + h_b$, it is possible to get the different uses of human capital in function of the effective tax rate, τ, and the human capital per capita, h:

$$\begin{aligned} h_y &= h_y(h,\tau) = (1-\tau)h && (33)\\ h_b &= h_b(\tau) = \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} && (34)\\ h_h &= h_h(h,\tau) = \tau h - \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} && (35) \end{aligned}$$

First, because taxes finance bureaucrats and teachers, a higher tax rate would involve less human capital devoted to producing goods. Thus, the human capital per capita dedicated to the production, h_y, is a decreasing function of the effective tax rate and an increasing function of human capital per capita. Second, the per capita number of bureaucrats is an increasing function of the effective tax rate since a higher effective tax rate requires a larger number of bureaucrats to implement it (see equation 31). Finally, the relationship between the per capita number of teachers, h_h, and the effective tax rate shows a hump-shape form. Two offsetting mechanisms generate this hump-shape: (i) a higher effective tax rate increases the government revenues that finance education, and (ii) a higher effective tax rate raises the bureaucratic cost.

Substituting equations (33) and (35) respectively in the production technology (equation 1) and the human capital accumulation equation (equation 2), we get the production of goods per capita and the human capital per capita accumulation equation as functions of the effective tax rate:

$$\begin{aligned} y(t) &= c(t) = c(h(t),\tau(t)) = B\left((1-\tau(t))h(t)\right)^{\alpha+\beta}\left(1-h(t)-s(t)\right)^{1-\alpha}\\ \dot{h}(t) &= h_h(h(t),\tau(t))^{\xi} s(t)^{1-\xi} - bh(t) \end{aligned}$$

[13]The reason for excluding transfers in the analysis is that transfers are inefficient in this economy. If there are transfers, additional bureaucrats would be required to collect necessary taxes to finance these transfers, which are costly and so well-being worsening. Note that because all households are alike, they pay the same amount of taxes and receive the same amount of transfers; consequently, their disposable income would not change in the case of costless tax collection. However, since tax collection is costly, introducing transfers would reduce the households' disposable income. Indeed, Households would pay taxes to finance the transfers they receive, but also the bureaucratic cost required.

where in the first equation we use the constraint that the per capita number of workers is equal to one: $1 = l(t) + h(t) + s(t)$.

The benevolent social planner's problem would be as follows:

$$
\begin{aligned}
&\max_{\{\tau(t),s(t)\}_{t=0}^{\infty}} \int_0^{\infty} \frac{\left(B\left((1-\tau(t))h(t)\right)^{\beta+\alpha}\left(1-h(t)-s(t)\right)^{1-\alpha}\right)^{1-\sigma}}{1-\sigma} e^{-(\rho-n)t}dt \\
&\dot{h}(t) \;=\; h_h(h(t),\tau(t))^{\xi} s(t)^{1-\xi} - bh(t) \qquad (36)\\
&h\left(0\right) \;>\; 0
\end{aligned}
$$

The first-order conditions (FOCs) of the optimization problem (36) are as follows:

$$
\begin{aligned}
\tfrac{(\alpha+\beta)[c(h(t),\tau(t))]^{1-\sigma}}{1-\tau(t)} &= \ell(t)\xi\left(\tfrac{s(t)}{h_h(h(t),\tau(t))}\right)^{1-\xi}\left(h(t)-\tfrac{1}{\gamma}\left(\tfrac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}}\right) && (37)\\
\tfrac{(1-\alpha)[c(h(t),\tau(t))]^{1-\sigma}}{1-h(t)-s(t)} &= \ell(t)(1-\xi)\left(\tfrac{h_h(h(t),\tau(t))}{s(t)}\right)^{\xi} && (38)\\
\tfrac{\left[\frac{\alpha+\beta}{h(t)}-\frac{1-\alpha}{1-h(t)-s(t)}\right][c(h(t),\tau(t))]^{1-\sigma}+\ell(t)\left[\xi\left(\frac{s(t)}{h_h(h(t),\tau(t))}\right)^{1-\xi}\tau(t)-m\right]}{\ell(t)} &+ \tfrac{\dot{\ell}(t)}{\ell(t)} = \rho && (39)
\end{aligned}
$$

where $\ell(t)$ is the Lagrange's multiplier, which is interpreted as the shadow price of human capital. The first equation, FOC (37), means that the marginal cost of taxation in terms of the present utility loss due to lower consumption, $\frac{(\alpha+\beta)[c(h(t),\tau(t))]^{1-\sigma}}{1-\tau(t)}$, should be equal to its marginal benefit, which is the value of the marginal increase in the future human capital due to the taxation, $\ell(t)\xi\left(\frac{s(t)}{h_h(h,\tau)}\right)^{1-\xi}\left(h(t)-\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}}\right)$. The reallocation of human capital from the production sector (where public revenues come from) to the education sector (where public expenditure takes place) generates these "marginal cost" and "marginal benefit" of taxation in respectively production and education. The second equation, FOC (38), means that the marginal cost of students in terms of the present utility loss due to lower consumption, $\frac{(1-\alpha)[c(h(t),\tau(t))]^{1-\sigma}}{1-h(t)-s(t)}$, should be equal to its marginal benefit, which consists of the marginal increase of the value of the future human capital due to students, $\ell(t)(1-\xi)\left(\frac{h_h(h(t),\tau(t))}{s(t)}\right)^{\xi}$. The reallocation of raw labor from production to education generates the "marginal cost" and the "marginal benefit" of students in production and education. Finally, FOC (39) means that the return of investing in human capital (left-hand side of the equation) should be equal to the discounted factor of the utility. The return of investing in human capital has two parts: the marginal net income of human capital divided by the price of the capital (generated in the production sector) and the capital gains (generated through human capital accumulation). The marginal net income includes: first, the marginal increase of the utility due to the future increase of production of goods generated by human capital, $\left[\frac{(\alpha+\beta)}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right][c(h(t),\tau(t))]^{1-\sigma}$; second, the value of the marginal product of future human capital, $\ell(t)\left[\xi\left(\frac{s(t)}{\tau(t)h(t)-\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}}\right)^{1-\xi}\tau(t)\right]$, minus the value of its depreciation, $\ell(t)m$. The marginal effect of investment in future production, $\left[\frac{(\alpha+\beta)}{h(t)} - \frac{1-\alpha}{1-h(t)-s(t)}\right]$, is equal to the increase of production due to more human capital, $\frac{\alpha}{h(t)}$, minus the decrease of production due to the reduction in raw labor that the increase in human capital requires, $-\frac{1-\alpha}{1-h(t)-s(t)}$. The value of its depreciation, $\ell(t)m$, consists of the value of the portion of skilled workers that die. The capital gains consist of the growth rate of the shadow price of human capital, $\frac{\dot{\ell}(t)}{\ell(t)}$.

Remark 10 *It follows from equation (37) that, at any optimal path, the marginal revenue of a higher tax rate, $h(t)$, should exceed its marginal cost (in terms of bureaucratic effort), $\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}}$. That is, $h(t) > \frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\tau(t)^{\frac{1-\gamma}{\gamma}} \Longleftrightarrow \tau(t) < (\Gamma)^{\frac{1}{1-\gamma}}(\gamma h(t))^{\frac{\gamma}{1-\gamma}}$. Thus, we will only consider tax rates that satisfy the above constraint.*

Using FOCs (37) and (38), it is possible to obtain the per capita number of students, s, as a function of human capital per capita, h, and the tax rate, τ (see the appendix):

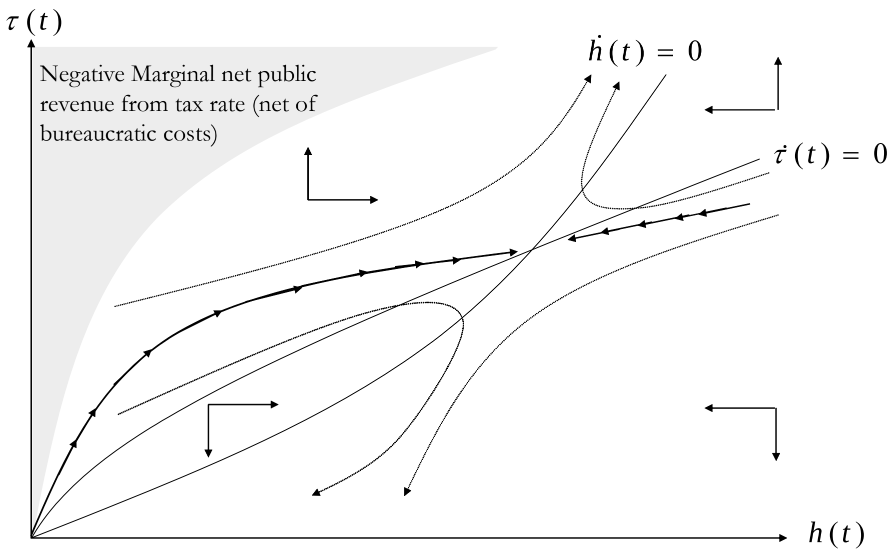
$$s = s(h,\tau) = \frac{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau} + \frac{\xi}{1-\xi}\left(h_h(h,\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \tag{40}$$

where $h_h(h,\tau)$ is the per capita number of teachers defined in equation (35). We observe that the per capita number of students is an increasing function of the tax rate for two reasons: (i) a higher tax rate implies more teachers, which reduces the cost of education (increases the probability of obtaining education). (ii) Furthermore, a higher tax rate involves less human capital in producing goods (see equation 33), making human capital scarcer and better paid. Moreover, the per capita number of students is a decreasing function of the human capital per capita since abundant human capital per capita reduces its marginal productivity and the return of education.

From this equation and the first-order conditions defined above, it is possible to derive a dynamic system of equations as a function of human capital per capita, h, and the tax rate, τ. Such a dynamic system is rather complicated and is in the appendix.

Proposition 11 *There is a non-empty subset of the parameter space such that a steady-state exists and is unique.*

Fig. 7.— Optimal tax rate dynamics



Proposition 12 *When the steady-state is unique, the steady-state is a saddle point.*

The above two propositions determine that for a subspace of the parameter space, the steady-state exists, is unique, and is a saddle point.

Figure 7 displays the dynamics toward the steady state. The shadow area represents combinations of human capital and tax rates in which the marginal cost of raising taxes exceeds its marginal revenue. Thus, the net marginal revenue from rising the tax rate is negative (considering the bureaucratic cost). Thus, these combinations are never efficient (see remark 10). We observe that human capital per capita and the tax rate increase along the transition when capital per capita is low. Consequently, the per capita number of teachers and bureaucrats increases along the transition (see equations 35 and 34). On the other hand, the per capita amount of human capital devoted to producing goods does not show a clear pattern. The increase of human capital per capita tends to raise the human capital dedicated to producing goods. Still, the rise in the tax rate has the opposite effect because it increases the number of teachers and bureaucrats (see equation 33). These results are in line with the ones in the benchmark model.

It follows from equations (33), (34), and (35) that the shares of human capital devoted to production, bureaucrats, and teachers are as follows:

$$\frac{h_y}{h} = (1-\tau) \tag{41}$$

$$\frac{h_b}{h} = \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\frac{1}{h} \tag{42}$$

$$\frac{h_h}{h} = \tau - \left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\frac{1}{h} \tag{43}$$

Figure 8 displays the evolution of the human capital share across sectors along the transition when we depart from a human capital per capita smaller than the steady-state level. Figure 8 is based on the above equations and the fact that the tax rate and the human capital per capita rise along the transition. We observe in figure 8 that: first, the share of teachers over total human capital, h_h/h, increases along the transition; second, the share of human capital devoted to production over total human capital, h_y/h, decreases and ; third, the behavior of the share of bureaucrats in total human capital, h_b/h, is ambiguous.

Thus, these results show that the increasing part of human capital devoted to public sector activities (bureaucracy and public education) at the expense of the private sector in the first stage of development is not a sign of lousy allocation of resources. On the contrary, it is consistent with the efficient allocation.

10. Public Sector Wage Premium

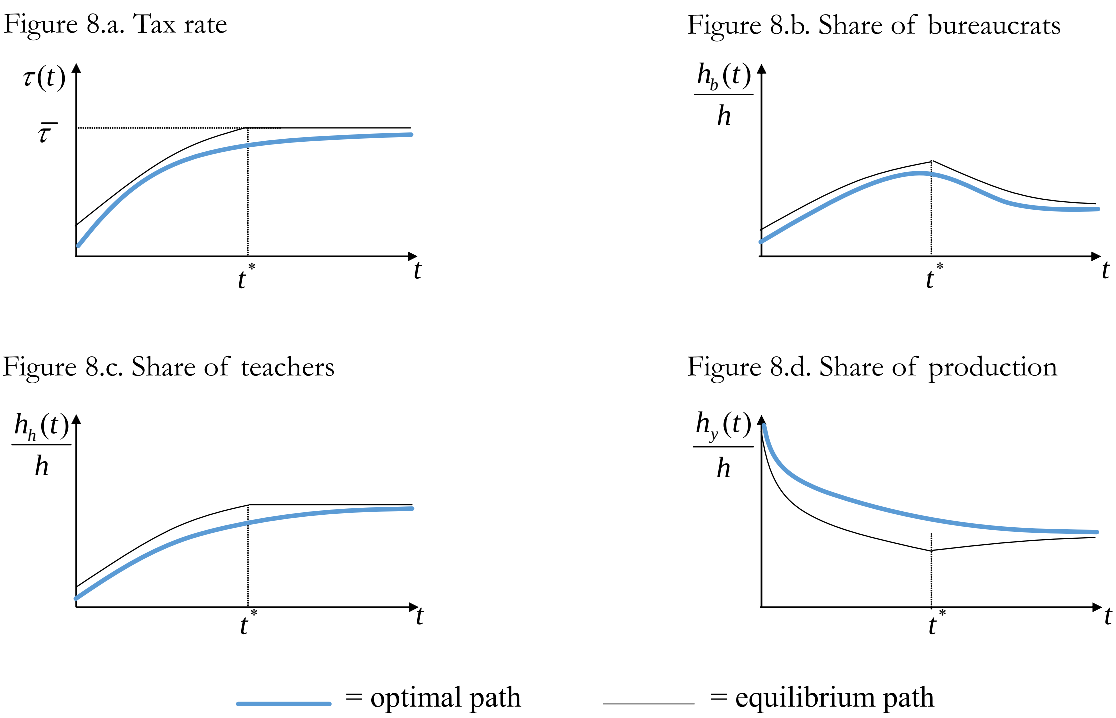
Some empirical papers, such as Depalo, Giordano, and Papapetrou (2015), have shown that wages in the public sector tend to be higher than in the private sector. We extend the model to introduce this observation.

Consider that the wage among public sector workers (both bureaucrats and teachers) is larger than the wage of skilled workers in the private sector (production). More precisely, consider that the after-tax wage premium in the public sector is equal to φw_h. To simplify, also assume that all skilled workers pay the same taxes, which are equal to τw_h. Therefore, the wage of skilled workers in the private sector is equal to w_h, the wage of skilled workers in the public sector is equal to $(1+\varphi)w_h$, and all skilled workers pay taxes equal to τw_h, where τ is given by equation (3). The government maximization problem (15) in this setting is as follows:

$$\max_{h_b} T(t) - w_h(t)(1+\varphi)h_b(t) \tag{44}$$

where the tax revenues, $T(t)$, are given by equation (16). Solving the model (using the tax

Fig. 8.— Equilibrium versus optimal allocation of human capital



rules of the benchmark model), we obtain:

$$\frac{h_b(h(t))}{h(t)} = \begin{cases} \left(\frac{\gamma\Gamma}{1+\varphi}\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \frac{\overline{h}_b}{h(t)} & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{45}$$

$$\frac{h_h(h(t))}{h(t)} = \begin{cases} \lambda\left(\frac{\gamma^{\gamma}\Gamma}{1+\varphi}\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{\gamma}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \frac{\lambda\overline{\tau}}{1+\varphi} & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{46}$$

$$h_y(h(t)) = \begin{cases} 1-(\gamma+\lambda)\left(\frac{\gamma^{\gamma}\Gamma}{1+\varphi}\right)^{\frac{1}{1-\gamma}} h(t)^{\frac{1}{1-\gamma}} & \text{if } h(t) < \overline{h} \\ \left(1-\frac{\lambda\overline{\tau}}{1+\varphi}\right) h(t) - \overline{h}_b & \text{if } h(t) \geq \overline{h} \end{cases}; \tag{47}$$

where $\overline{h} = \frac{(1+\varphi)\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}$ and $\overline{h}_b = \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}$. Let's compare the above equations with the ones in the benchmark model (equations 23, 24 and 25). We conclude that the fact that the public sector wages have a premium does not affect the evolution of human capital allocation. It only produces a level effect. Thus, this public wage premium does not affect any of our paper's results.

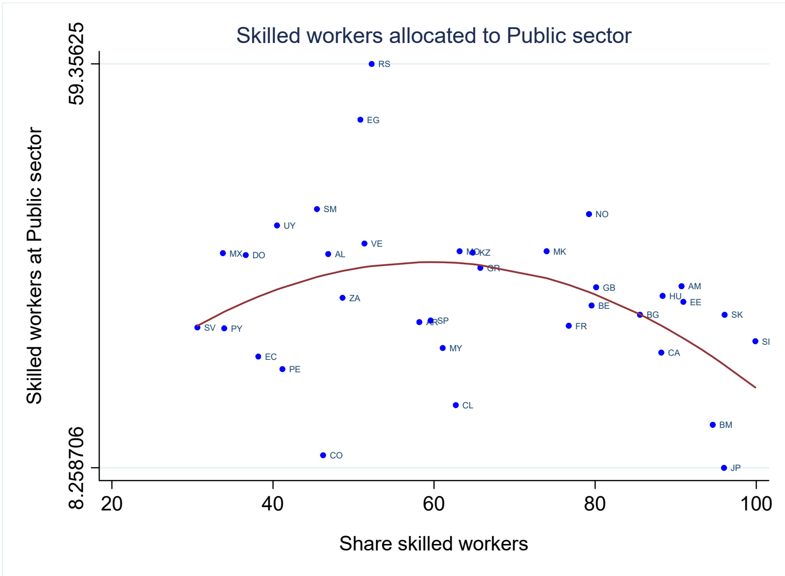
11. Implications of the model and empirical evidence

Here we discuss two main implications of the proposed theory in the light of the empirical evidence available: the hump-shaped relationship between the human capital allocated to the public sector (education and bureaucracy) and the amount of human capital and the relationship between human capital and institutional quality.

11.1. Public sector along the transition: hump-shaped pattern

The model predicts a hump-shaped pattern in the share of human capital allocated to the public sector (bureaucracy plus teachers) as countries accumulate human capital. In the first stage of development, when human capital rises, it becomes less scarce. Consequently, the government hires more bureaucrats who then implement a higher effective tax rate and more teachers to produce more human capital. The number of teachers and bureaucrats grows faster than skilled workers in the economy, implying an increasing share of human capital allocated to the public sector. Once the statutory tax rate is reached, the tax rate remains steady, independently of the human capital per capita (see figure 3.a). Thus, the government only hires the number of bureaucrats needed to collect the statutory tax rate

Fig. 9.— Share of human capital at the public sector



and hires teachers at the same rhythm as the human capital increases. Consequently, the share of human capital allocated to the public sector (bureaucrats plus teachers) decreases when human capital per capita rises (see figures 3.b and 3.c).

Figure 9 shows the relationship between human capital accumulation and the fraction of skilled workers allocated to the public sector for a broad sample of developing and developed countries. The share of skilled workers in the public sector is obtained from the International Labor Office (ILO) data set of 2009. It is measured as the share of public sector employment over total skilled workers. The public sector employment, provided by ILO, covers all employments of the general government sector as defined in System of National Accounts 1993 plus employment of publicly owned enterprises and companies, resident and operating at local; state (or regional); and central levels of government. It also covers all individuals employed directly by those institutions, without regard for the particular type of employment contract. The number of skilled workers is calculated as the total number of workers with advanced and intermediate education. According to the International Standard Classification of Education (ISCED 97), intermediate education includes Upper secondary education and Post-secondary non-tertiary education. Advanced education comprises the first stage of tertiary education (not leading directly to an advanced research qualification).

We observe a non-linear relationship between the share of skilled workers allocated to the public sector and the share of skilled workers in the population. We have also estimated the impact of human capital accumulation in determining the proportion of skilled workers allocated to the public sector. The best resulting fit is the non-linear one:

$$PublicShare = \beta_1 HumanCapitalShare + \beta_2 HumanCapitalShare^2 \tag{48}$$

The model has been estimated using ordinary least squares. Table 1 shows the results of the estimation. The sample size is about 35 countries (developed and developing ones).

Table 1: Bureaucracy and human capital share

	Coef.	Std.Err.	t	$p > \|t\|$
HKS	1.154	0.103	11.21	0.00
HKS^2	-0.009	0.001	-4.46	0.00
			N	34
			R^2	91.2

Figure 9 also shows the fitted values of the model. These results confirm the predictions of the model. In addition, we observe a hump-shaped form between the share of human capital allocated to the bureaucratic sector and the human capital level in the economy. Estimation coefficients show that the linear term has a positive effect, whereas the square term has a negative sign. Though the sample size is reduced, the model predicts very well the observed data.

11.2. Public sector and institutional quality

Our theory shows that an improvement in the technology of the bureaucracy (institutional enhancement) generates a reallocation of human capital from bureaucracy to the private sector, which reduces the wage of skilled workers and discourages the accumulation of human capital. Finally, the economy reaches a higher level of GDP but a lower level of human capital. This feature implies a positive relationship between institutional quality and GDP per capita and a negative relationship between institutional quality and human capital per capita and human capital allocated to the public sector.

To test this finding, we have estimated the impact of the "bureaucracy effectiveness" in determining the size of the public sector, measured as the share of human capital allocated to the public sector. More precisely, we test the robustness of the hump-shaped relationship documented in the previous subsection to institutional quality changes. We now estimate the following equation using ordinary least square:

$$PublicShare = \beta_1 HumanCapitalShare + \beta_2 HumanCapitalShare^2 + \beta_3 IntsitutQuality \tag{49}$$

Institutional quality (InstitQuality) is proxied by three different measures: "Government effectiveness", "Rule of law"and "Control of corruption", provided by the Worldwide Governance Indicators (World Bank). "Government effectiveness"(GE) takes into account perceptions of the quality of policy formulation and implementation; the quality of the civil service; the degree of its independence from political pressures and the quality of public services; and the credibility of the government's commitment to such policies. "Rule of law"(ROL) considers perceptions of the extent to which individuals have confidence by the rules of society, and in particular, the quality of the police and the courts; property rights and contract enforcement; and the likelihood of violence and crime. "Control of corruption"(CC) considers perceptions of the extent to which public power is exercised for private gain, including both grand and petty forms of corruption, and capture of the state by private interest and elites. These three measures range from -2.5 (weak) to 2.5 (strong) governance performance. We also take the observations of these variables in 2009.

Table 2 shows theestimation results when we use the three different measures of institutional quality.

We observe that the hump-shaped relation between the share of the bureaucracy and the human capital level in the economy is robust to the institutional quality effect. Moreover, we also observe a negative relationship between the fraction of the skilled labor allocated to the public sector and the institutional quality. Finally, although the significance level is low, we observe that signs of the relationships are robust to the three measures we use (government effectiveness, the rule of law, and control of corruption).

12. Conclusion

This paper builds a theory explaining how human capital allocation evolves during development. Such a theory sheds light on the human capital allocation among different activities and public and private sectors when economies grow. It also contributes to understanding the weak and controversial empirical relationship between human capital and economic growth. We build a model in which the public education system is essential to human capital accumulation. To finance public education, the government needs to hire skilled workers as bureaucrats that collect taxes. Thus, the government absorbs part of

Table 2: Human capital at public sector and institutional quality

	Coef.	Std.Err.	t	$p > \|t\|$
HK	1.095	0.112	9.76	0.00
HKS^2	-0.008	0.001	-5.69	0.00
GE	-2.625	2.594	-1.01	0.31
			N	34
			R^2	91.3
HK	1.090	0.118	9.23	0.00
HKS^2	-0.008	0.001	-5.57	0.00
ROL	-1.82	2.19	-0.83	0.41
			N	34
			R^2	91.3
HK	1.089	0.112	9.67	0.00
HKS^2	-0.008	0.001	-5.87	0.00
CC	-2.334	2.11	-1.10	0.27
			N	34
			R^2	91.2

the economy's human capital: it employs bureaucrats to collect taxes and teachers for the public education system. When human capital per capita is low, the scarcity of human capital prevents the government from hiring enough bureaucrats to implement the statutory tax rate. The resulting low effective tax rate implies that it is impossible to employ many teachers, and consequently, most human capital is used to produce goods (in the private sector). Along the transition, human capital rises, making human capital more abundant and allowing the government to hire more bureaucrats to implement a higher effective tax rate and hire more teachers for the public education system. The fact that an increasing part of human capital is recruited by the government during the development process, diverting human capital from the private sector, may involve a slowdown of production in the private sector and GDP per capita. However, this does not necessarily mean that the government's absorption of human capital is wasteful or inefficient. On the contrary, we analyzed the optimal human capital allocation and showed that the efficient allocation follows the same pattern as in the benchmark model. Indeed, in the efficient allocation, a growing part of human capital is absorbed by the public sector at the expense of the private sector, implying an increasing optimal tax rate along the transition.

The paper emphasizes the crucial role that institutional differences play in understanding the weak effect of human capital on macroeconomic performance documented in the literature. Differences in institutions across countries may generate a misleading negative correlation between human capital and economic performance at the macro level. The reason is that countries with poor institutions require more bureaucrats, raising the incentive to invest in education and increasing the amount of human capital. However, poor institutions reduce the steady-state levels of production of goods and GDP. Thus, countries with weaker institutions would have more human capital and less GDP than countries with stronger institutions. As a result, a misleading negative correlation between GDP and human capital may arise among countries with different levels of institutional quality.

Thus, our paper sheds some light on the empirical weak and controversial relationship between human capital and economic growth from two different perspectives: first, the diversion of human capital to the public sector in the first stage of development that involves a low impact of human capital on production and GDP and; second, differences in institutions across countries that may generate a misleading negative correlation between GDP and human capital.

We have tested empirically two of the predictions of our model. The first is the hump-shaped relationship between the share of skilled workers in the public sector and the percentage of skilled workers over the total labor force and, the second is the negative effect of institutional quality on the share of skilled workers employed by the government. We showed that the empirical evidence supports both predictions of the model.

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14. Appendix

Proof of Proposition 2:

Using the definition of GDP and equations (11) and (10):

$$gdp = wl + w_h h = (1-\alpha)y + \frac{\alpha y}{h_y}h =$$
$$= \left[(1-\alpha) + \alpha\frac{h}{h_y}\right] y = \frac{(1-\alpha)\left(\frac{h_y}{h}\right) + \alpha}{\left(\frac{h_y}{h}\right)^{1-\alpha-\beta}} B h^{\alpha+\beta} l^{1-\alpha} \tag{50}$$

The derivative of the above expression with respect to (h_y/h) is as follows:

$$
\begin{aligned}
&\frac{\partial gdp}{\partial\left(\frac{h_y}{h}\right)} = \\
&gdp\frac{(1-\alpha)\left[\alpha+\beta\right]\left(\frac{h_y}{h}\right)-(1-\alpha-\beta)\alpha}{\left[(1-\alpha)\left(\frac{h_y}{h}\right)+\alpha\right]\left(\frac{h_y}{h}\right)} \geq \\
&gdp\frac{(1-\alpha)\left[\alpha+\beta\right](1-(\gamma+\lambda)\overline{\tau})-(1-\alpha-\beta)\alpha}{\left[(1-\alpha)\left(\frac{h_y}{h}\right)+\alpha\right]\left(\frac{h_y}{h}\right)}
\end{aligned}
$$

where in the inequality we have used the fact that the minimum ratio (h_y/h) is equal to $1-(\gamma+\lambda)\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}}\overline{h}^{\frac{-\gamma}{1-\gamma}} = 1-(\gamma+\lambda)\overline{\tau}$ (see equation 25). Thus, the following sufficient condition implies that the derivative $\frac{\partial gdp}{\partial\left(\frac{h_y}{h}\right)}$ is positive:

$$
\begin{aligned}
&(1-\alpha)\left[\alpha+\beta\right](1-(\gamma+\lambda)\overline{\tau})-(1-\alpha-\beta)\alpha \geq 0 \Leftrightarrow \\
&\beta \geq \frac{\alpha(\gamma+\lambda)\overline{\tau}}{\left[1-(\gamma+\lambda)\overline{\tau}\right]+\alpha(\gamma+\lambda)\overline{\tau}}(1-\alpha) \equiv \underline{\beta}
\end{aligned}
$$

Proof of Proposition 3

It follows from (50) that:

$$
gdp(h,s) = \frac{(1-\alpha)\left(\frac{h_y(h)}{h}\right)+\alpha}{\left(\frac{h_y(h)}{h}\right)^{1-\alpha-\beta}}Bh^{\alpha+\beta}(1-s-h)^{1-\alpha}
$$

where $h_y(h)$ is defined in equation (22). If $h<\overline{h}$ then:

$$
\begin{aligned}
&\frac{\frac{\partial gdp(h,s)}{\partial h}}{gdp} = \frac{(1-\alpha)\left[\alpha+\beta\right]\left(\frac{h_y(h)}{h}\right)-(1-\alpha-\beta)\alpha}{\left[(1-\alpha)\left(\frac{h_y(h)}{h}\right)+\alpha\right]\left(\frac{h_y(h)}{h}\right)}\frac{\partial\left(\frac{h_y(h)}{h}\right)}{\partial h}+\frac{(\alpha+\beta)}{h}-\frac{(1-\alpha)}{1-s-h} = \\
&\left[-\frac{(1-\alpha)\left[\alpha+\beta\right]\left(\frac{h_y(h)}{h}\right)-(1-\alpha-\beta)\alpha}{\left[(1-\alpha)\left(\frac{h_y(h)}{h}\right)+\alpha\right]}\frac{\gamma}{1-\gamma}\frac{1-\frac{h_y(h)}{h}}{\frac{h_y(h)}{h}}+(\alpha+\beta)\right]\frac{1}{h}-\frac{(1-\alpha)}{1-s-h} =
\end{aligned}
$$

$$\frac{\frac{\partial gdp\left(\overline{h},s\right)}{\partial h}}{gdp}=$$

$$-\left[\tfrac{(1-\alpha)[\alpha+\beta](1-(\gamma+\lambda)\overline{\tau})-(1-\alpha-\beta)\alpha}{(1-\alpha)(1-(\gamma+\lambda)\overline{\tau})+\alpha}\tfrac{\gamma}{1-\gamma}\tfrac{(\gamma+\lambda)\overline{\tau}}{1-(\gamma+\lambda)\overline{\tau}}-(\alpha+\beta)\right]\tfrac{\Gamma^{\frac{1}{\gamma}}}{\overline{\tau}^{\frac{1}{\gamma}}}-\tfrac{(1-\alpha)}{1-s-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}=$$

$$-\left[\left[(\alpha+\beta)-\tfrac{1}{(1-\alpha)(1-(\gamma+\lambda)\overline{\tau})+\alpha}\right]\tfrac{\gamma}{1-\gamma}\tfrac{(\gamma+\lambda)\overline{\tau}}{1-(\gamma+\lambda)\overline{\tau}}-(\alpha+\beta)\right]\tfrac{\Gamma^{\frac{1}{\gamma}}}{\overline{\tau}^{\frac{1}{\gamma}}}-\tfrac{(1-\alpha)}{1-s-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}=$$

$$-\left[\left[(\alpha+\beta)\left[1-\tfrac{1-\gamma}{\gamma}\tfrac{1-(\gamma+\lambda)\overline{\tau}}{(\gamma+\lambda)\overline{\tau}}\right]-\tfrac{1}{(1-\alpha)(1-(\gamma+\lambda)\overline{\tau})+\alpha}\right]\tfrac{\gamma}{1-\gamma}\tfrac{(\gamma+\lambda)\overline{\tau}}{1-(\gamma+\lambda)\overline{\tau}}\right]\tfrac{\Gamma^{\frac{1}{\gamma}}}{\overline{\tau}^{\frac{1}{\gamma}}}-\tfrac{(1-\alpha)}{1-s-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}=$$

$$-\left[\left[\tfrac{(\alpha+\beta)}{\gamma}\left[1-\tfrac{1-\gamma}{(\gamma+\lambda)\overline{\tau}}\right]-\tfrac{1}{1-(1-\alpha)(\gamma+\lambda)\overline{\tau}}\right]\tfrac{\gamma}{1-\gamma}\tfrac{(\gamma+\lambda)\overline{\tau}}{1-(\gamma+\lambda)\overline{\tau}}\right]\tfrac{\Gamma^{\frac{1}{\gamma}}}{\overline{\tau}^{\frac{1}{\gamma}}}-\tfrac{(1-\alpha)}{1-s-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}$$

Note that:

$$\lim_{\gamma\to 1}\lim_{\alpha\to 1}\frac{(\alpha+\beta)}{\gamma}\left[1-\frac{1-\gamma}{(\gamma+\lambda)\overline{\tau}}\right]-\frac{1}{1-(1-\alpha)(\gamma+\lambda)\overline{\tau}}=\beta>$$

Thus, taken as given the other parameters, it follows from continuity that there is a surounding of $(\alpha,\gamma)=(1,1)$ (a subset of the parameter space) such that:

$$\frac{\frac{\partial gdp\left(\overline{h},s\right)}{\partial h}}{gdp}<0$$

In such subset of the parameter space, it follows also from continuity that, there is $\varepsilon>0$ such that if $h\in\left(\overline{h}-\varepsilon,\overline{h}\right)$ then $\frac{\partial gdp(h,s)}{\partial h}<0$.

Dynamic behavior: Dynamic system of the economy

The Hamiltonian associated to optimization problem (4) may be written as follows:

$$u\left(w_h(t)\left(1-\tau(t)\right)h(t)+w(t)\left(1-h(t)-s(t)\right)+tr(t)\right)e^{-(\rho-n)t}+\ell(t)e^{-(\rho-n)t}\left[\mu(t)s\left(t\right)-bh\left(t\right)\right]$$

The first-order conditions are as follows:

$$c(t)^{-\sigma}w(t)=\ell(t)\mu(t)\Rightarrow\ell(t)=\frac{c(t)^{-\sigma}w(t)}{\mu(t)}=c(t)^{-\sigma}p_h(t)\tag{51}$$

$$\dot{\lambda}(t)-(\rho-n)\ell(t)=-c(t)^{-\sigma}\left[w_h(t)\left(1-\tau(t)\right)-w(t)\right]+\lambda b\tag{52}$$

Note that we have not derived with respect to $\mu(t)$ since this is an "aggregate" variable that does not depend on the decision of an individual household. Using the above equations we obtain:

$$\begin{aligned}
\frac{\dot{\ell}(t)}{\ell(t)} &= -\sigma\frac{\dot{c}(t)}{c(t)} + \frac{\dot{p_h}(t)}{p_h(t)} \Rightarrow \\
\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\sigma}\left[\frac{w_h(t)\left(1-\tau(t)\right)-w(t)}{p_h(t)} + \frac{\dot{p_h}(t)}{p_h(t)} - (\rho+m)\right]
\end{aligned}$$

where we have used the fact that $n = b - m$. Using equations. (10) and (11), the goods market clear condition, and the definition of $\mu(t)$ it follows that:

$$\begin{aligned}
&\left[\frac{\xi}{s(t)} + \frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)}\right]\dot{s}(t) = \\
&-\left(\frac{h_h(t)}{s(t)}\right)^{\xi}\left[\frac{\alpha\left(1-\tau(t)\right)\left(1-h(t)-s(t)\right)-(1-\alpha)\,h_y(t)}{(1-\alpha)\,h_y(t)}\right] + (\rho+m) + \\
&+\left[\frac{\xi}{h_h(t)}\frac{\partial h_h(h(t))}{\partial h} - \frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)} + \frac{(\alpha+\beta)(\sigma-1)}{h_y(t)}\frac{\partial h_y(h(t))}{\partial h}\right]\dot{h}(t)
\end{aligned}$$

Thus, the dynamic system that defines the dynamic behavior of the economy is as follows:

$$\begin{aligned}
&\left[\tfrac{\xi}{s(t)} + \tfrac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)}\right]\dot{s}(t) = \\
&-\left(\tfrac{h_h(t)}{s(t)}\right)^{\xi}\left[\tfrac{\alpha(1-\tau(t))(1-h(t)-s(t))-(1-\alpha)h_y(t)}{(1-\alpha)h_y(t)}\right] + (\rho+m) + \\
&+\left[\tfrac{\xi}{h_h(t)}\tfrac{\partial h_h(h(t))}{\partial h} - \tfrac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s(t)} + \tfrac{(\alpha+\beta)(\sigma-1)}{h_y(t)}\tfrac{\partial h_y(h(t))}{\partial h}\right]\underbrace{\left[\left(h_h(h(t))\right)^{\xi} s(t)^{1-\xi} - bh(t)\right]}_{\dot{h}(t)}
\end{aligned} \tag{53}$$

$$\dot{h}(t) = \left(h_h(h(t))\right)^{\xi} s(t)^{1-\xi} - bh(t) \tag{54}$$

Proof of Proposition 6

The dynamic system (53)-(54) implies that the following two equations should hold at the steady-state:

$$\left.\begin{aligned}
\left(\tfrac{h_h(h)}{s}\right)^{\xi}\left[\tfrac{\alpha(1-\tau)(1-h-s)-(1-\alpha)h_y}{(1-\alpha)h_y}\right] &= (\rho+m) \\
h_h(h)^{\xi}s^{1-\xi} &= bh
\end{aligned}\right\} \Rightarrow \Psi(h) = 0$$

where $\Psi(h)$ is the function that defines the steady-state:

$$\Psi(h) = \left(\frac{h_h(h)}{bh}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\left(1-\tau(h)\right)\left(1-h\left(1+b\left(\frac{bh}{h_h(h)}\right)^{\frac{\xi}{1-\xi}}\right)\right)-(1-\alpha)\,h_y}{(1-\alpha)\,h_y(h)}\right] - (\rho+m)$$

Function $\Psi(h)$ has two branches.

If $h \leq \overline{h}$:

$$\Psi(h) =$$

$$\left(\frac{\lambda(\gamma^{\gamma}\Gamma)^{\frac{\gamma}{1-\gamma}}}{b}\right)^{\frac{\xi}{1-\xi}} h^{\frac{\gamma\xi}{(1-\gamma)(1-\xi)}} \times$$

$$\left[\frac{\alpha\left(1-\left(\gamma\Gamma^{\frac{1}{\gamma}}h\right)^{\frac{\gamma}{1-\gamma}}\right)\left(1-h\left(1+b\left(\frac{bh}{\lambda(\gamma^{\gamma}\Gamma h)^{\frac{1}{1-\gamma}}}\right)^{\frac{\xi}{1-\xi}}\frac{1}{h^{\frac{\gamma\xi}{(1-\gamma)(1-\xi)}}}\right)\right)-(1-\alpha)\left[h-(\gamma+\lambda)(\gamma^{\gamma}\Gamma h)^{\frac{1}{1-\gamma}}\right]}{(1-\alpha)\left(h-(\gamma+\lambda)(\gamma^{\gamma}\Gamma h)^{\frac{1}{1-\gamma}}\right)}\right]$$

$$-(\rho+m) =$$

$$\frac{\left(\frac{\lambda(\gamma^{\gamma}\Gamma)^{\frac{\gamma}{1-\gamma}}}{b}\right)^{\frac{\xi}{1-\xi}}}{h^{\frac{1-\gamma-\xi}{(1-\gamma)(1-\xi)}}}\left[\frac{\alpha\left(1-(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h^{\frac{\gamma}{1-\gamma}}\right)}{(1-\alpha)\left(1-(\gamma+\lambda)(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h^{\frac{\gamma}{1-\gamma}}\right)}\left(1-h-b\left(\frac{b}{\lambda(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}}\right)^{\frac{\xi}{1-\xi}}h^{\frac{1-\gamma-\xi}{(1-\gamma)(1-\xi)}}\right)-h\right]$$

$$-(\rho+m) \tag{55}$$

If $h \geq \overline{h}$:

$$\Psi(h) = \left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]-(\rho+m) \tag{56}$$

It follows from equations (55) and (56) that if $\gamma+\xi \leq 1$ then $\Psi(h)$ is strictly decreasing and, consequently, if there is steady-state then it is unique. In order to have a steady-state h^{ss} such that $h^{ss} > \overline{h}$, the following condition should hold:

$$\Psi(\overline{h}) = \left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}\left(1+b\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}{\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}(1-\overline{\tau}(\lambda+\gamma))}-1\right]-(\rho+m) > 0 \Leftrightarrow$$

$$\Gamma > \overline{\Gamma} \equiv \left[\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}\left[\left(1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right)(1-\overline{\tau}(\lambda+\gamma))+\left(1+b\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\right]}{\gamma}\right]^{\gamma}$$

Finally, note that for h close enough to $\left(1+b\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)^{-1}$, the function $\Psi(h)$ becomes negative. Thus, if $\Gamma > \overline{\Gamma}$ we can guarantee that there is a unique steady-state and $h^{ss} > \overline{h}$ at such steady-state. ■

Proof of Proposition 7

The dynamic system (53)-(54) when $h(t) > \overline{h}$ is as follows:

$$\begin{aligned} \dot{h}\,(t) &= F_h\,(h(t), s(t)) \qquad (57) \\ \dot{s}\,(t) &= F_s\,(h(t), s(t)) \qquad (58) \end{aligned}$$

where:

$$\begin{aligned} F_h\,(h,s) &= (\lambda\overline{\tau}h)^{\xi}\, s^{1-\xi} - bh \\ F_s\,(h,s)\left[\frac{\xi}{s} + \frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right] &= \\ -\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}&\left[\frac{\alpha\,(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s}{(1-\overline{\tau}\lambda)\,h - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right] + (\rho+m) + \\ +\left[\frac{\xi}{h} + \frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right.&\left.\left[\frac{(\alpha+\beta)(\sigma-1)\,(1-\overline{\tau}\lambda)}{1+(\sigma-1)(1-\alpha)}\frac{1-h-s}{(1-\overline{\tau}\lambda)\,h - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right]\right] F_h\,(h,s) \end{aligned}$$

First, we will prove that the locus $\dot{s}\,(t) = F_s\,(h(t), s(t)) = 0$ exists. Note that:

$$\begin{aligned} &\frac{\partial F_s\,(h^{ss}, s^{ss})}{\partial s} = \\ &\left[\frac{\xi}{s^{ss}} + \frac{1+(\sigma-1)(1-\alpha)}{1-h^{ss}-s^{ss}}\right]^{-1} \times \left\{ (\rho+m)\left[\frac{\xi}{s^{ss}} + \frac{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1}{(1-\overline{\tau}\lambda)h^{ss} - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h^{ss} - \frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1}\right] + \right. \\ &\left. + \left[\frac{\xi}{h^{ss}} + \frac{1+(\sigma\text{-}1)(1\text{-}\alpha)}{1-h^{ss}-s^{ss}}\left[\frac{(\alpha+\beta)(\sigma\text{-}1)\,(1\text{-}\overline{\tau}\lambda)}{1+(\sigma\text{-}1)(1\text{-}\alpha)}\frac{1-h^{ss}-s^{ss}}{(1\text{-}\overline{\tau}\lambda)\,h^{ss}\text{-}\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}} - 1\right]\right](1\text{-}\xi)\left(\frac{\lambda\overline{\tau}h^{ss}}{s^{ss}}\right)^{\xi}\right\} > 0 \end{aligned}$$

Thus, it follows from the Implicit Function Theorem that in a surrounding of the steady-state it is possible to define $s^{\dot{s}=0}(h) \Leftrightarrow F_s\left(h, s^{\dot{s}=0}(h)\right) = 0$.

Secondly, we will prove that the locus $\dot{s}\,(t) = F_s\,(h(t), s(t)) = 0$ is above the locus $\dot{h}\,(t) = F_h\,(h(t), s(t)) = 0$ when $h < h^{ss}$. Let's define $s^{\dot{h}=0}(h) \Leftrightarrow F_h\left(h, s^{\dot{h}=0}(h)\right) = 0 \Leftrightarrow s^{\dot{h}=0}(h) = \left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}} h$. Note that $F_s\left(h, s^{\dot{h}=0}(h)\right)\left[\frac{\xi}{s^{\dot{h}=0}(h)} + \frac{1+(\sigma-1)(1-\alpha)}{1-h(t)-s^{\dot{h}=0}(h)}\right] = -\,\Psi(h)$ (see equation 56). Thus, it follows from the proof of proposition **6** that

$$\left\{ \begin{array}{l} \text{If } h < h^{ss} \Rightarrow F_s\left(h, s^{\dot{h}=0}(h)\right) < 0 \\ \text{If } h > h^{ss} \Rightarrow F_s\left(h, s^{\dot{h}=0}(h)\right) > 0 \end{array} \right\} \tag{59}$$

Now, we need to prove that in a surrounding of the steady-state, when $h < h^{ss}$, for some $s(t) > s^{\dot{h}=0}(h)$ it is possible to obtain that $F_s\left(h(t), s(t)\right) > 0$. Note that:

$$\frac{F_s\left(h,s\right)}{F_h\left(h,s\right)}\left[\frac{\xi}{s}+\frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right] = \tag{60}$$

$$\frac{1}{F_h\left(h,s\right)}\left\{-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha\left(1-\overline{\tau}\right)}{\left(1-\alpha\right)}\frac{1-h-s}{\left(1-\overline{\tau}\lambda\right)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]+\left(\rho+m\right)\right\}+$$

$$+\left[\frac{\xi}{h}+\frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\left[\frac{(\alpha+\beta)(\sigma-1)\left(1-\overline{\tau}\lambda\right)}{1+(\sigma-1)(1-\alpha)}\frac{1-h-s}{\left(1-\overline{\tau}\lambda\right)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]\right] \tag{61}$$

It follows from the fact that when $h < h^{ss}$ then $F_h\left(h, s^{ss}\right) > 0$, that if $\frac{F_s(h,s^{ss})}{F_h(h,s^{ss})} > 0$ when $h < h^{ss}$ then $F_s\left(h(t), s(t)\right) > 0$ when $s(t) = s^{ss}$. This implies that there is $s^{\dot{s}=0}(h) \in \left(s^{\dot{h}=0}(h), s^{ss}\right)$. Note that the first term of equation (61) is always finite:

$$\lim_{h\to h^{ss}}\frac{-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]+\left(\rho+m\right)}{F_h\left(h,s^{ss}\right)} =$$

$$\lim_{h\to h^{ss}}\frac{\frac{\partial\left[-\left(\frac{\lambda\overline{\tau}h}{s}\right)^{\xi}\left[\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right]+(\rho+m)\right]}{\partial h}}{\frac{\partial F_h(h,s^{ss})}{\partial h}}$$

$$\frac{\left(\rho+m\right)\left[-\frac{\xi}{h^{ss}}+\frac{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\left[\frac{1}{1-h^{ss}-s^{ss}}+\frac{(1-\overline{\tau}\lambda)}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\right]}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\frac{1-h^{ss}-s^{ss}}{(1-\overline{\tau}\lambda)h-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1}\right]}{-(1-\xi)b}$$

Thus, given that the first term of equation (61) is finite, it is easy to check (see equation 61) that $\lim\limits_{\sigma\to+\infty} \frac{F_s(h^{ss},s^{ss})}{F_h(h^{ss},s^{ss})}\left[\frac{\xi}{s^{ss}}+\frac{1+(\sigma-1)(1-\alpha)}{1-h^{ss}-s^{ss}}\right] = +\infty$ (remark, note that the steady-state variable values do not depend on σ). Thus, there is $\overline{\sigma}$, such that if $\sigma > \overline{\sigma}$ then $\frac{F_s(h^{ss},s^{ss})}{F_h(h^{ss},s^{ss})}\left[\frac{\xi}{s^{ss}}+\frac{1+(\sigma-1)(1-\alpha)}{1-h^{ss}-s^{ss}}\right] > 0$ and $\left[\frac{\xi}{s^{ss}}+\frac{1+(\sigma-1)(1-\alpha)}{1-h^{ss}-s^{ss}}\right] > 0$. Thus, if $\sigma > \overline{\sigma}$, in a surrounding of h^{ss}, $(h^{ss}-\varepsilon, h^{ss}+\varepsilon)$:

$$\frac{F_s\left(h,s^{ss}\right)}{F_h\left(h,s^{ss}\right)}\left[\frac{\xi}{s}+\frac{1+(\sigma-1)(1-\alpha)}{1-h-s}\right] > 0 \Rightarrow$$
$$\left\{\begin{array}{l}\text{If } h < h^{ss} \Rightarrow F_h\left(h,s^{ss}\right) > 0 \Rightarrow F_s\left(h,s^{ss}\right) > 0 \\ \text{If } h > h^{ss} \Rightarrow F_h\left(h,s^{ss}\right) < 0 \Rightarrow F_s\left(h,s^{ss}\right) < 0\end{array}\right\} \tag{62}$$

Equations. (59) and (62) imply that:

$$\left\{\begin{array}{l}\text{If } h < h^{ss} \Rightarrow s^{\dot{s}=0}\left(h\right) \in \left(s^{\dot{h}=0}\left(h\right), s^{ss}\right) \\ \text{If } h > h^{ss} \Rightarrow s^{\dot{s}=0}\left(h\right) \in \left(s^{ss}, s^{\dot{h}=0}\left(h\right)\right)\end{array}\right. \tag{63}$$

This implies that:

$$\left\{\begin{array}{l}\text{If } h < h^{ss} \Rightarrow s^{\dot{s}=0}\left(h\right) - s^{\dot{h}=0}\left(h\right) > 0 \\ \text{If } h > h^{ss} \Rightarrow s^{\dot{s}=0}\left(h\right) - s^{\dot{h}=0}\left(h\right) < 0\end{array}\right\} \Rightarrow$$
$$\frac{\partial\left[s^{\dot{s}=0}\left(h^{ss}\right) - s^{\dot{h}=0}\left(h^{ss}\right)\right]}{\partial h} < 0 \Rightarrow$$
$$\frac{\partial s^{\dot{s}=0}\left(h^{ss}\right)}{\partial h} - \frac{\partial s^{\dot{h}=0}\left(h^{ss}\right)}{\partial h} < 0 \tag{64}$$

The dynamic system in a surrounding of the steady-state may be linearized as follows:

$$\begin{bmatrix}\dot{h}(t)\\ \dot{s}(t)\end{bmatrix} = \begin{bmatrix}\frac{\partial F_h(h^{ss},s^{ss})}{\partial h} & \frac{\partial F_h(h^{ss},s^{ss})}{\partial s}\\ \frac{\partial F_s(h^{ss},s^{ss})}{\partial h} & \frac{\partial F_s(h^{ss},s^{ss})}{\partial s}\end{bmatrix}\begin{bmatrix}h(t)-h^{ss}\\ s(t)-s^{ss}\end{bmatrix}$$

Eigenvalues are defined as follows:

$$\begin{vmatrix}\frac{\partial F_h(h^{ss},s^{ss})}{\partial h}-\lambda & \frac{\partial F_h(h^{ss},s^{ss})}{\partial s}\\ \frac{\partial F_s(h^{ss},s^{ss})}{\partial h} & \frac{\partial F_s(h^{ss},s^{ss})}{\partial s}-\lambda\end{vmatrix} = \lambda^2 - tr\,\lambda + Det = 0 \Rightarrow \lambda = \frac{tr \pm \sqrt{tr^2-4Det}}{2}$$

It follows from (64) that:

$$\begin{aligned}
tr &= \frac{\partial F_h\left(h^{ss},s^{ss}\right)}{\partial h}+\frac{\partial F_s\left(h^{ss},s^{ss}\right)}{\partial s} \\
Det &= \underbrace{\frac{\partial F_h\left(h^{ss},s^{ss}\right)}{\partial h}}_{-}\underbrace{\frac{\partial F_s\left(h^{ss},s^{ss}\right)}{\partial s}}_{+}-\underbrace{\frac{\partial F_h\left(h^{ss},s^{ss}\right)}{\partial s}}_{+}\frac{\partial F_s\left(h^{ss},s^{ss}\right)}{\partial h}= \\
Det &= \frac{\partial F_s\left(h^{ss},s^{ss}\right)}{\partial s}\frac{\partial F_h\left(h^{ss},s^{ss}\right)}{\partial s}\left[\frac{\frac{\partial F_h(h^{ss},s^{ss})}{\partial h}}{\frac{\partial F_h(h^{ss},s^{ss})}{\partial s}}-\frac{\frac{\partial F_s(h^{ss},s^{ss})}{\partial h}}{\frac{\partial F_s(h^{ss},s^{ss})}{\partial s}}\right]= \\
Det &= \underbrace{\frac{\partial F_s\left(h^{ss},s^{ss}\right)}{\partial s}}_{+}\underbrace{\frac{\partial F_h\left(h^{ss},s^{ss}\right)}{\partial s}}_{+}\underbrace{\left[\frac{\partial s^{\dot{s}=0}(h^{ss})}{\partial h}-\frac{\partial s^{\dot{h}=0}(h^{ss})}{\partial h}\right]}_{-}<0
\end{aligned}$$

Thus, one of the eigenvalues is positive and the other is negative. This means that the steady-state is a saddle point. Furthermore, it follows from (63) that $s(t)$ is increasing when $h(t)<h^{ss}$ and decreasing when $h(t)>h^{ss}$.

Proof of Proposition 8

From equation (30) it is easy to see that the skill premium evolves according to the following equation:

$$\dot{sp}(t)=-\frac{\alpha}{1-\alpha}\frac{1-h(t)-s(t)}{h_y(t)}\left[\frac{\dot{h}_y(t)}{h_y(t)}+\frac{\dot{h}(t)+\dot{s}(t)}{1-h(t)-s(t)}\right] \tag{65}$$

It follows from the above equation that if $h(t)<h^{ss}$ and $\dot{h}_y>0$, then the skill premium is always decreasing if $\dot{h}_y(t)\geq 0$. It is follows from (22) that $\dot{h}_y(t)$ is always positive except in the interval $\left[\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}},\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}\right]$ (remember that $\overline{h}\equiv\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}$). Thus, we will focus the proof in such interval, in which $\dot{sp}$ is as follows:

$$\dot{sp}=-\frac{\alpha}{1-\alpha}\frac{1-h-s}{h_y}\left[\frac{\left[1\text{-}\frac{(\gamma+\lambda)}{1-\gamma}\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}}h(t)^{\frac{\gamma}{1-\gamma}}+\frac{h(t)\text{-}(\gamma+\lambda)(\gamma^{\gamma}\Gamma)^{\frac{1}{1-\gamma}}h(t)^{\frac{1}{1-\gamma}}}{1-h-s}\right]\dot{h}}{h_y}+\frac{\dot{s}}{1-h-s}\right]$$

Thus, the following condition is sufficient in order that the skill premium is always decreasing (we have already proveen in proposition **7** that $\dot{s}>0$ when $s(t)<s^{ss}$):

$$f(\overline{\tau})=\min_{h\in\left[\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}}\frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}},\frac{\overline{\tau}^{\frac{1-\gamma}{\gamma}}}{\gamma\Gamma^{\frac{1}{\gamma}}}\right]}\left[1-\frac{(\gamma+\lambda)}{1-\gamma}\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}}h^{\frac{\gamma}{1-\gamma}}+\frac{h\text{-}(\gamma+\lambda)\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}}h^{\frac{1}{1-\gamma}}}{1-h}\right]\geq 0 \tag{66}$$

The above function $f(\tau)$ is decreasing in τ. Let's define $\widehat{\tau}$ as follows:

$$\widehat{\tau} = \begin{cases} 1 & \text{if } f(1) \geq 0 \\ \max\left\{\widehat{\tau} \text{ such that } \forall\overline{\tau} \leq \widehat{\tau},\ f(\overline{\tau}) \geq 0\right\} & \text{if } f(1) < 0 \end{cases}$$

Note that:

$$f\left(\frac{1-\gamma}{\gamma+\lambda}\right) = \min_{h \in \left[\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}, \left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}\right]} \left[1 - \frac{(\gamma+\lambda)}{1-\gamma}\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h^{\frac{\gamma}{1-\gamma}} + \frac{h\text{-}(\gamma+\lambda)\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} h^{\frac{1}{1-\gamma}}}{1-h}\right]$$

$$\left[1 - \frac{(\gamma+\lambda)}{1-\gamma}\left(\gamma^{\gamma}\Gamma\right)^{\frac{1}{1-\gamma}} \left(\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{1-\gamma}} + \frac{h_y\left(\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}\right)}{1 - \left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}}\right] =$$

$$\frac{h_y\left(\left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}\right)}{1 - \left(\frac{1-\gamma}{\gamma+\lambda}\right)^{\frac{1-\gamma}{\gamma}} \frac{1}{\gamma\Gamma^{\frac{1}{\gamma}}}} > 0$$

Thus, $\widehat{\tau} > \frac{1-\gamma}{\gamma+\lambda}$.

Proof of Proposition 9

It follows from (56) and (22) that:

$$
\begin{aligned}
&\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\left(1-\overline{\tau}\right)}{\left(1-\alpha\right)}\frac{1-h^{ss}\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]}{\left(1-\overline{\tau}\lambda\right)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}-1\right] \;=\; (\rho+m) \Leftrightarrow \\
&\frac{\alpha\left(1-\overline{\tau}\right)}{\left(1-\alpha\right)}\left[1-h^{ss}\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\right] \;=\; \left[1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\left[\left(1-\overline{\tau}\lambda\right)h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}\right] \\
&h^{ss} \;=\; \frac{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}+\left[1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]+\left[1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\left(1-\overline{\tau}\lambda\right)} \\
&h^{ss}_y \;=\; (1\text{-}\overline{\tau}\lambda)\,h^{ss}-\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}} \;=\; \frac{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\left[(1\text{-}\overline{\tau}\lambda)-\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}\right]}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]+\left[1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\left(1-\overline{\tau}\lambda\right)} \\
&\frac{\frac{\partial h^{ss}}{\partial\Gamma}}{h_{ss}} \;=\; -\frac{\left[1+(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}{\frac{\alpha(1-\overline{\tau})}{(1-\alpha)}+\left[(\rho+m)\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}+1\right]\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\frac{1}{\gamma\Gamma} \;<\; 0 \\
&\frac{\frac{\partial h^{ss}_y}{\partial\Gamma}}{h^{ss}_y} \;=\; \frac{\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}{(1\text{-}\overline{\tau}\lambda)-\left[1+b\left(\frac{b}{\lambda\overline{\tau}}\right)^{\frac{\xi}{1-\xi}}\right]\frac{\overline{\tau}^{\frac{1}{\gamma}}}{\Gamma^{\frac{1}{\gamma}}}}\frac{1}{\gamma\Gamma} \;>\; 0
\end{aligned}
$$

At the steady-state (see equation 56)

$$
\begin{aligned}
\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\left(1-\overline{\tau}\right)}{1-\alpha}\left(\frac{l}{h^{ss}_y}\right)-1\right] \;&=\; m+\rho \Leftrightarrow \frac{l^{ss}}{h^{ss}_y} \;=\; \frac{1+\frac{(m+\rho)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}}{\frac{\alpha(1-\overline{\tau})}{1-\alpha}} \\
y^{ss} \;=\; \left(h^{ss}_y\right)^{\alpha+\beta}\left(l^{ss}\right)^{1-\alpha} \;&=\; B\left(\frac{1+\frac{(m+\rho)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}}{\frac{\alpha(1-\overline{\tau})}{1-\alpha}}\right)^{1-\alpha}\left(h^{ss}_y\right)^{1+\beta}
\end{aligned}
$$

Thus:

$$
\frac{\frac{\partial y^{ss}}{\partial\Gamma}}{y^{ss}} = (1+\beta)\,\frac{\frac{\partial h^{ss}_y}{\partial\Gamma}}{h^{ss}_y} > 0 \tag{67}
$$

Note that at the steady-state (see equation 56):

$$\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}\left[\frac{\alpha\left(1-\overline{\tau}\right)}{1-\alpha}\left(\frac{1-\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)h^{ss}}{h_y^{ss}}\right)-1\right]=m+\rho\Leftrightarrow$$

$$\frac{1}{h_y^{ss}}-\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\frac{h^{ss}}{h_y^{ss}}=\frac{1+\frac{(m+\rho)}{\left(\frac{\lambda\overline{\tau}}{b}\right)^{\frac{\xi}{1-\xi}}}}{\frac{\alpha(1-\overline{\tau})}{1-\alpha}}\Rightarrow$$

$$-\frac{1}{h_y^{ss}}\frac{\frac{\partial h_y^{ss}}{\partial\Gamma}}{h_y^{ss}}-\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)\frac{\partial\left(\frac{h^{ss}}{h_y^{ss}}\right)}{\partial\Gamma}=0\Leftrightarrow\frac{\partial\left(\frac{h^{ss}}{h_y^{ss}}\right)}{\partial\Gamma}=-\frac{\frac{1}{h_y^{ss}}\frac{\frac{\partial h_y^{ss}}{\partial\Gamma}}{h_y^{ss}}}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}\tag{68}$$

The GDP per capita is as follows (see 50):

$$gdp=y\left[(1-\alpha)+\alpha\left(\frac{h^{ss}}{h_y^{ss}}\right)\right]\Leftrightarrow\ln gdp=\ln y+\ln\left[(1-\alpha)+\alpha\left(\frac{h^{ss}}{h_y^{ss}}\right)\right]$$

Using equations (67) and (68), it follows that at the steady-state:

$$\frac{\frac{\partial gdp^{ss}}{\partial\Gamma}}{gdp^{ss}}=\frac{\frac{\partial y^{ss}}{\partial\Gamma}}{y^{ss}}+\frac{\alpha}{\left[(1-\alpha)+\alpha\left(\frac{h^{ss}}{h_y^{ss}}\right)\right]}\frac{\partial\left(\frac{h^{ss}}{h_y^{ss}}\right)}{\partial\Gamma}$$

$$\frac{\frac{\partial gdp^{ss}}{\partial\Gamma}}{gdp^{ss}}=(1+\beta)\frac{\frac{\partial h_y^{ss}}{\partial\Gamma}}{h_y^{ss}}-\frac{\alpha}{\left[(1-\alpha)+\alpha\left(\frac{h^{ss}}{h_y^{ss}}\right)\right]}\frac{\frac{1}{h_y^{ss}}\frac{\frac{\partial h_y^{ss}}{\partial\Gamma}}{h_y^{ss}}}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}=$$

$$\left[(1+\beta)-\frac{\alpha}{\left[(1-\alpha)h_y^{ss}+\alpha h\right]}\frac{1}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}\right]\frac{\frac{\partial h_y^{ss}}{\partial\Gamma}}{h_y^{ss}}>$$

$$\left[1+\beta-\frac{\alpha}{\left(1+\left(\frac{b}{(\lambda\overline{\tau})^{\xi}}\right)^{\frac{1}{1-\xi}}\right)}\right]\frac{\frac{\partial h_y^{ss}}{\partial\Gamma}}{h_y^{ss}}>0$$

where in the last inequality we use the fact that $h_y<h<1$. Thus, since $\frac{\partial h_y^{ss}}{\partial\Gamma}>0$ then $\frac{\partial gdp^{ss}}{\partial\Gamma}>0$.

Optimal Fiscal Policy

The Hamiltonian of problem (36) is as follows:

$$\begin{aligned} H \;=\; & \frac{\left(B\left((1-\tau(t))h(t)\right)^{\alpha+\beta}\left(1-h(t)-s(t)\right)^{1-\alpha}\right)^{1-\sigma}}{1-\sigma} e^{-(\rho-n)t} + \\ & \ell(t)e^{-(\rho-n)t}\left[\left(\tau(t)h(t)-\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{\xi} s(t)^{1-\xi} - bh(t)\right] \end{aligned}$$

The first-order conditions of optimization problem (36) are as follows:

$$\frac{(\alpha+\beta)\left[c(h(t),\tau(t))\right]^{1-\sigma}}{1-\tau(t)} \;=\; \ell(t)\xi\frac{\dot{h}(t)+bh(t)}{h_h(h,\tau)}\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau(t)} \tag{69}$$

$$\frac{(1-\alpha)\left[c(h(t),\tau(t))\right]^{1-\sigma}}{1-h(t)-s(t)} \;=\; \ell(t)(1-\xi)\frac{\dot{h}(t)+bh(t)}{s(t)} \tag{70}$$

$$\begin{aligned} & \dot{\ell}(t)-(\rho-n)\ell(t) \;= \\ & -\left[c(h(t),\tau(t))\right]^{1-\sigma}\left[\frac{\alpha+\beta}{h(t)}-\frac{1-\alpha}{1-h(t)-s(t)}\right]-\ell(t)\left[\xi\frac{\dot{h}(t)+bh(t)}{h_h(h,\tau)}\tau(t)-b\right] \end{aligned} \tag{71}$$

where $\dot{h}(t)+bh(t)=\left(h_h(h(t),\tau(t))\right)^{\xi}s(t)^{1-\xi}$ and $h_h(h,\tau)=\tau h-\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}$. Using (69) and (70), it yields:

$$s(t) = \frac{\frac{\alpha+\beta}{1-\alpha}h_h(h(t),\tau(t))\frac{\tau(t)}{1-\tau(t)}\left(1-h(t)\right)}{\frac{\alpha+\beta}{1-\alpha}h_h(h(t),\tau(t))\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \tag{72}$$

Using (37) and (72), it yields:

$$\begin{aligned} \ell(t) \;=\; & \frac{\left(\frac{1}{B}\right)^{\sigma-1}\frac{(\alpha+\beta)}{\xi}\frac{\tau(t)^{\xi}}{(1-\tau(t))^{\xi+(\sigma-1)(\alpha+\beta)}}\left(\frac{1-\alpha}{\alpha+\beta}\right)^{1-\xi}\left(\frac{1-\xi}{\xi}\right)^{(\sigma-1)(1-\alpha)}}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{1+(\sigma-1)(1-\alpha)}} \times \\ & \left(\frac{\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\right]h_h(h(t),\tau(t))-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}}{(1-h(t))}\right)^{1-\xi+(\sigma-1)(1-\alpha)} \left(\frac{1}{h(t)}\right)^{(\alpha+\beta)(\sigma-1)} \end{aligned} \tag{73}$$

Differentiation the above equation with respect to time it follows that:

$$
\begin{aligned}
&\frac{\dot{\ell}(t)}{\ell(t)} = \Omega_1(h(t),\tau(t))\frac{\dot{\tau}(t)}{\tau(t)} \\
&-\Bigg\{ \frac{\left((1-\xi+(\sigma-1)(1-\alpha))\right)\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)\left(\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\right]h_h(h(t),\tau(t))-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \\
&+\frac{\xi\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\tau}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)h_h(h,\tau)} \\
&+\left[(\sigma-1)-\frac{\xi}{h_h(h,\tau)}\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau(t)}\frac{(1-\tau)}{\alpha+\beta}\right]\left[\frac{\alpha+\beta}{h(t)}-\frac{1-\alpha}{1-h(t)-s(t)}\right]-\frac{1-\xi}{(1-h(t))}\Bigg\}\dot{h} \qquad (74)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Omega_1(h,\tau)=\frac{\frac{\partial\ell}{\partial\tau}}{\ell(t)}=\frac{\xi+(\sigma-1)(\alpha+\beta)}{1-\tau}+ \\
&\Bigg[\frac{\Omega_2(h,\tau)\left(\frac{1-\gamma}{\gamma}\frac{1}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)\left(\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\right]h_h(h(t),\tau(t))-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \\
&+\left(1-\xi+\left(\sigma-1\right)\left(1-\alpha\right)\right)\frac{\left[\frac{\alpha+\beta}{1-\alpha}\frac{1}{(1-\tau(t))^2}\right]\left[h_h(h(t),\tau(t))\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\right]h_h(h(t),\tau(t))-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}}\Bigg]\tau>0 \\
&\Omega_2(h,\tau)= \\
&\left[\left(1+\left(\sigma-1\right)\left(1-\alpha\right)\right)\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}\right]+\frac{\xi^2}{1-\xi}\right]\left[h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]+ \\
&\left[\left(1\text{-}\xi+\left(\sigma\text{-}1\right)\left(1\text{-}\alpha\right)\right)\frac{\alpha\gamma}{1-\alpha}\left(\frac{1-\tau(t)(1-\tau(t))}{(1-\tau(t))^2}\right)+\left(1+\left(\sigma\text{-}1\right)\left(1\text{-}\alpha\right)\right)\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1\text{-}\tau(t)}\right]\right]\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}
\end{aligned}
$$

Using (39), (73) and (74), it yields the dynamic system that determine the dynamic behavior of the economy

$$
\begin{aligned}
\dot{h}(t) &= \left(h_h(h(t),\tau(t))\right)^{\xi}\left(s(h(t),\tau(t))\right)^{1-\xi} - bh(t) \qquad (75)\\
\Omega_1(h(t),\tau(t))\frac{\dot{\tau}(t)}{\tau(t)} &= \\
& F^h(h(t),\tau(t))\left\{\frac{((1-\xi+(\sigma-1)(1-\alpha)))\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right]\tau}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)\left(\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau(t)}{1-\tau(t)}+\frac{\xi}{1-\xi}\right]h_h(h(t),\tau(t))-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}\right.\\
&+\frac{\xi\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\tau}{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)h_h(h,\tau)}\\
&\left.+\left[(\sigma\text{-}1)-\frac{\xi}{h_h(h,\tau)}\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau(t)}\frac{(1-\tau)}{\alpha+\beta}\right]\left[\frac{\alpha+\beta}{h(t)}-\frac{1-\alpha}{1-h(t)-s(t)}\right]-\frac{1-\xi}{(1-h(t))}\right\}\\
&-\xi\frac{bh(t)}{h_h(h,\tau)}\left[\frac{\left(h_h(h(t),\tau(t))-\frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{(\alpha+\beta)\frac{\tau(t)}{(1-\tau(t))}}\left[\frac{\alpha+\beta}{h(t)}-\frac{1-\alpha}{1-h(t)-s(t)}\right]+\tau(t)\right]+\rho+m \qquad (76)
\end{aligned}
$$

where

$$
\begin{aligned}
F^h(h,\tau) &= \left(h_h(h,\tau)\right)^{\xi}\left(s(h,\tau)\right)^{1-\xi} - bh\\
s(h,\tau) &= \frac{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}
\end{aligned}
$$

Proof of Proposition 11

It follows from the dynamic system (75) and (76) that at the steady-state the following two equation should hold simultaneously:

$$
G^{hss}(h,\tau) = \frac{(h_h(h,\tau))^{\xi}}{h}\left(\frac{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}\right)^{1-\xi} - b = 0 \qquad (77)
$$

$$
G^{\tau ss}(h,\tau) = \frac{\xi bh}{h_h(h,\tau)}\left[\tau+\left[\frac{\alpha+\beta}{h}-\frac{1\text{-}\alpha}{1\text{-}h\text{-}s(h,\tau)}\right]\frac{1}{\alpha+\beta}\frac{\left(h_h(h,\tau)\text{-}\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}}\right] - (\rho+m) = 0 \qquad (78)
$$

Remark 13 *It follows from the above equations that the steady-state values do not depend on σ.*

Lemma 14 *There is a function $h^{\dot{h}=0}(\tau)$ such that $G^{hss}(h^{\dot{h}=0}(\tau),\tau) = 0$ and $\frac{\partial h^{\dot{h}=0}(\tau)}{\partial\tau} > 0$.*

Proof. Note that:

$$\frac{\partial\left(\frac{(h_h(h,\tau))^{\xi}}{h}\right)}{\partial h} = \frac{(h_h(h,\tau))^{\xi}}{h^2}\left(\xi\frac{\tau h}{h_h(h,\tau)}-1\right) < \frac{(h_h(h,\tau))^{\xi}}{h^2}\left(\xi\frac{\frac{1}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}-1\right) =$$
$$= \frac{(h_h(h,\tau))^{\xi}}{h^2}\left(\frac{\xi}{1-\gamma}-1\right) \leq 0 \tag{79}$$

where in the first inequality we use the fact that $\tau h \geq \frac{1}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}} \Leftrightarrow h \geq \left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\frac{\tau^{\frac{1-\gamma}{\gamma}}}{\gamma}$ and the fact that the function $\frac{h}{h_h(h,\tau)}$ is decreasing[14] in h, and in the last inequality we have use the assumption that $\xi \leq 1-\gamma$. Furthermore:

$$\frac{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} = \frac{\frac{\alpha+\beta}{1-\alpha}\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha+\beta}{1-\alpha}\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(1-\frac{\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{h_h(h,\tau)}\right)} \tag{80}$$

The above function is clearly decreasing in h. Therefore, it follows from (77), (79) and (80) that:

$$\frac{\partial G^{hss}(h,\tau)}{\partial h} < 0$$

Now we analyze the derivative of $G^{hss}(h,\tau)$ with respect to τ:

$$\frac{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}(1-h)}{\frac{\alpha+\beta}{1-\alpha}h_h(h,\tau)\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} = \frac{\frac{\alpha+\beta}{1-\alpha}(1-h)}{\frac{\alpha+\beta}{1-\alpha}+\frac{\xi}{1-\xi}\frac{\left(1-\frac{\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{h_h(h,\tau)}\right)}{\frac{\tau}{1-\tau}}}$$

Note that:

$$\frac{\partial\left(\frac{\tau^{\frac{1}{\gamma}}}{h_h(h,\tau)}\right)}{\partial\tau} = \frac{\frac{1}{\gamma}\tau^{\frac{1}{\gamma}}h_h(h,\tau)-\tau^{\frac{1}{\gamma}}\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau\left(h_h(h,\tau)\right)^2} = \frac{\frac{1-\gamma}{\gamma}\tau^{\frac{1}{\gamma}}\left(h_h(h,\tau)+\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau\left(h_h(h,\tau)\right)^2} > 0$$

Therefore $G^{hss}(h,\tau)$ is clearly increasing in τ:

$$\frac{\partial G^{hss}(h,\tau)}{\partial\tau} > 0$$

---

[14] $\frac{\partial\left(\frac{h}{h_h(h,\tau)}\right)}{\partial h} = \frac{1}{h_h(h,\tau)}\left[1-\frac{\tau h}{\tau h-\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}\right] < 1$

Thus, it is possible to define $h^{\dot{h}=0}(\tau) \Leftrightarrow G^{hss}(h^{\dot{h}=0}(\tau),\tau) = 0$. It follows from the Implicit function Theorem that:

$$\frac{\partial h^{\dot{h}=0}(\tau)}{\partial \tau} = -\frac{\overbrace{\frac{\partial G^{hss}(h,\tau)}{\partial \tau}}^{\oplus}}{\underbrace{\frac{\partial G^{hss}(h,\tau)}{\partial h}}_{\ominus}} > 0$$

■

The following lemma will prove a statement that is related with $G^{\tau ss}(h,\tau)$.

Lemma 15 $\lim\limits_{\tau\to 0} \frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$

Proof. There are two possible cases:
1) If $\liminf\limits_{\tau\to 0} \frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)} > 0$ then $\lim\limits_{\tau\to 0} \frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$.
2) If $\liminf\limits_{\tau\to 0} \frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)} = 0$. Then there is a sequence $\{\tau_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to+\infty} \tau_i = 0$ and $\lim\limits_{i\to+\infty} \frac{\left(h_h(h^{\dot{h}=0}(\tau_i),\tau_i) - \frac{1-\gamma}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau_i),\tau_i)} = 0$. Then

$$\lim_{i\to+\infty} \frac{\left(h_h(h^{\dot{h}=0}(\tau_i),\tau_i) - \frac{1-\gamma}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau_i),\tau_i)} = \lim_{i\to+\infty} \frac{\left(\tau_i h^{\dot{h}=0}(\tau_i) - \frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau_i h^{\dot{h}=0}(\tau_i) - \left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}} =$$

$$\lim_{i\to+\infty} \frac{\left(\tau_i h^{\dot{h}=0}(\tau_i) - \frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau_i h^{\dot{h}=0}(\tau_i)(1-\gamma) + \gamma\left(\tau_i h^{\dot{h}=0}(\tau_i) - \frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} =$$

$$\lim_{i\to+\infty} \frac{\left(1 - \frac{\frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\tau_i h^{\dot{h}=0}(\tau_i)}\right)}{1-\gamma+\gamma\left(1 - \frac{\frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\tau_i h^{\dot{h}=0}(\tau_i)}\right)} = 0 \Rightarrow \tag{81}$$

$$\lim_{i\to+\infty} \frac{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}{h^{\dot{h}=0}(\tau_i)} = 1 \Rightarrow \tag{82}$$

Note that:

$$\lim_{i\to+\infty}\left(\frac{\left(h_h(h^{\dot{h}=0}(\tau_i),\tau)\right)^{\xi}}{bh^{\dot{h}=0}(\tau)}\right)^{\frac{1}{1-\xi}} =$$

$$\lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i h^{\dot{h}=0}(\tau_i)-\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{h^{\dot{h}=0}(\tau_i)}\right)^{\xi}\left(h^{\dot{h}=0}(\tau_i)\right)^{1-\xi}\right)^{\frac{1}{1-\xi}} =$$

$$\lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i\frac{h^{\dot{h}=0}(\tau_i)}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}-\frac{\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}}{\frac{h^{\dot{h}=0}(\tau_i)}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}}\right)^{\xi}\left(\frac{h^{\dot{h}=0}(\tau_i)}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}\right)^{1-\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i 1-\frac{\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}}{1}\right)^{\xi}(1)^{1-\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\tau_i\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}-\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}}\right)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}\left(\frac{\frac{1-\gamma}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}}{\frac{1}{\gamma}\left(\frac{\tau_i}{\Gamma}\right)^{\frac{1}{\gamma}}\frac{1}{\tau_i}}\right)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}(\tau_i)^{\frac{1-\gamma}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}$$

$$= \lim_{i\to+\infty}\left(\frac{1}{b}(1-\gamma)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}\tau_i^{\frac{\xi}{1-\xi}-\frac{1-\gamma}{\gamma}}$$

$$= \lim_{i\to+\infty}\frac{\left(\frac{1}{b}(1-\gamma)^{\xi}\left(\frac{1}{\gamma}\left(\frac{1}{\Gamma}\right)^{\frac{1}{\gamma}}\right)^{1-\xi}\right)^{\frac{1}{1-\xi}}}{(\tau_i)^{\frac{1-\gamma-\xi}{\gamma}}} = +\infty$$

where in the third inequality we use (77). Thus, it follows from equation (77) that:

$$\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)} =$$

$$\frac{1-\xi}{\xi}\frac{\alpha+\beta}{1-\alpha}\left[\left(\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau)\right)^{\xi}}{bh^{\dot{h}=0}(\tau)}\right)^{\frac{1}{1-\xi}}\left(1-h^{\dot{h}=0}(\tau)\right)-1\right] \Rightarrow$$

$$\lim_{i\to+\infty}\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)} = +\infty$$

where we used equation (82) in the forth equality. Thus, in the two possible cases mentioned above:

$$\lim_{\tau\to 0}\frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$$

■

Now we proceed to prove proposition 11:

Proof. At steady-state the following equation should hold (see equations (77), (78), and lemma 14):

$$G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) =$$
$$\xi b\left[\frac{\tau h^{\dot{h}=0}(\tau)}{h_h(h^{\dot{h}=0}(\tau),\tau)} + \left[(\alpha+\beta) - \frac{(1-\alpha)h^{\dot{h}=0}(\tau)}{1-h^{\dot{h}=0}(\tau)-s(h^{\dot{h}=0}(\tau),\tau)}\right]\frac{1}{\alpha+\beta}\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)}\right]$$
$$-(\rho+m) = 0$$

Let's define $f(\tau) = \xi b\left[\frac{\tau h^{\dot{h}=0}(\tau)}{h_h(h^{\dot{h}=0}(\tau),\tau)} + \left[(\alpha+\beta) - \frac{(1-\alpha)h^{\dot{h}=0}(\tau)}{1-h^{\dot{h}=0}(\tau)-s(h^{\dot{h}=0}(\tau),\tau)}\right]\frac{1}{\alpha+\beta}\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\frac{\tau}{1-\tau}h_h(h^{\dot{h}=0}(\tau),\tau)}\right]$.

Note that $G^{hss}(1,\tau) = -b$ (see equation 77), therefore $h^{\dot{h}=0}(\tau) < 1$. Thus, $f(\tau)$ is a continuous function for $\tau \in (0,1]$. Furthermore, it follows from lemma 15 that $\lim_{\tau\to 0} f(\tau) = +\infty$. Thus, if $(\rho+m) \geq \min_{\tau\in[0,1]} f(\tau)$ then there is at least a τ^{ss} such that $f(\tau^{ss}) = \rho+m$. Since $\lim_{\tau\to 0}\frac{\frac{\left(h_h(h^{\dot{h}=0}(\tau),\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{h_h(h^{\dot{h}=0}(\tau),\tau)}}{\frac{\tau}{1-\tau}} = +\infty$, $f(\tau)$ is strictly decreasing in an interval $(0,\hat{\tau})$ where $\hat{\tau} > 0$. Thus, if $(\rho+m) > \max_{\tau\in[\hat{\tau},1]} f(\tau)$, then $f(\tau^{ss}) = \rho+m$ for a $\tau^{ss} \in (0,\hat{\tau})$. Since in such interval $f(\tau)$ is strictly decreasing, there is a unique τ^{ss} such that $f(\tau^{ss}) = \rho+m$. ■

Proof of Proposition 12

It is easy to check that $F^h(h,\tau)$ is increasing in τ and decreasing in h, thus, it follows from the Implicit Function Theorem that the locus $\dot{h}(t) = 0$ has a positive slope:

$$\tau_{\dot{h}=0}(h) \overset{def}{\Leftrightarrow} F^h\left(h, \tau_{\dot{h}=0}(h)\right) = 0; \quad \frac{\partial \tau_{\dot{h}=0}(h)}{\partial h} = -\frac{\frac{\partial F^h(h,\tau)}{\partial h}}{\frac{\partial F^h(h,\tau)}{\partial \tau}} > 0$$

Let's define $F^\tau(h,\tau)$ as the function that determine $\dot{\tau}(t)$ (see equation 76)

$$\begin{aligned} F^\tau(h,\tau) &= \frac{\tau}{\Omega_1(h,\tau)} \times \\ &\left\{ \left[\left(\frac{((1-\xi+(\sigma-1)(1-\alpha)))\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau}{1-\tau}\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right]}{\left(\left[\frac{\alpha+\beta}{1-\alpha}\frac{\tau}{1-\tau}+\frac{\xi}{1-\xi}\right]h_h(h,\tau)-\frac{\xi}{1-\xi}\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} + \frac{\xi\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}}{h_h(h,\tau)} \right) \frac{\tau}{\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)} \right.\right. \\ &+ \left.\left((\sigma\text{-}1) - \frac{\xi}{h_h(h,\tau)} \frac{\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{\tau} \frac{(1-\tau)}{\alpha+\beta} \right) \left[\frac{\alpha+\beta}{h} - \frac{1-\alpha}{1-h-s(h,\tau)}\right] - \frac{1-\xi}{(1-h)} \right] F^h(h,\tau) \\ &\left. -\xi\frac{bh}{h_h(h,\tau)} \left[\frac{\left(h_h(h,\tau)-\frac{1-\gamma}{\gamma}\left(\frac{\tau}{\Gamma}\right)^{\frac{1}{\gamma}}\right)}{(\alpha+\beta)\frac{\tau}{(1-\tau)}} \left[\frac{\alpha+\beta}{h} - \frac{1-\alpha}{1-h-s}\right] + \tau \right] + \rho + m \right\} \end{aligned} \tag{83}$$

Lemma 16 *In a surrounding of h^{ss} if $h < h^{ss}$ then $\tau_{\dot{h}=0}(h) < \tau_{\dot{\tau}=0}(h)$ and if $h > h^{ss}$ then $\tau_{\dot{h}=0}(h) > \tau_{\dot{\tau}=0}(h)$. Furthermore, if $\tau(t) < \tau_{\dot{\tau}=0}(h(t))$ then $\tau_{\dot{\tau}=0}(h(t)) > 0$ and if $\tau(t) > \tau_{\dot{\tau}=0}(h(t))$ then $\tau_{\dot{\tau}=0}(h(t)) > 0$. The locus $\tau_{\dot{\tau}=0}(h(t))$ has positive slope($\frac{\partial \tau_{\dot{h}=0}(h)}{\partial h} > 0$). Finally, $\frac{\partial F^\tau(h^{ss},\tau^{ss})}{\partial \tau} > 0$.*

Proof. Consider a point (h,τ) in the locus $\dot{h} = 0$ $(\tau = \tau_{\dot{h}=0}(h))$, it follows from (76) and the definition of $F^h(h,\tau)$ that:

$$F^\tau\left(h, \tau_{\dot{h}=0}(h)\right) \frac{\Omega_1(h, \tau_{\dot{h}=0}(h))}{\tau_{\dot{h}=0}(h)} = -G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) = -G^{\tau ss}(h^{\dot{h}=0}(\tau), \tau) \tag{84}$$

We prove already in the proof of proposition 11 that:

$$\begin{aligned} G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) > 0 \text{ if } \tau < \tau^{ss} &\Leftrightarrow G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) > 0 \text{ if } h < h^{ss} \\ G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) = 0 \text{ if } \tau = \tau^{ss} &\Leftrightarrow G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) = 0 \text{ if } h = h^{ss} \\ G^{\tau ss}(h^{\dot{h}=0}(\tau),\tau) < 0 \text{ if } \tau > \tau^{ss} &\Leftrightarrow G^{\tau ss}(h, \tau_{\dot{h}=0}(h)) < 0 \text{ if } h > h^{ss} \end{aligned}$$

where we have used the fact that $h^{\dot{h}=0}(\tau)$ a strictly increasing and therefore bijective function. Given equation (84) this implies that:

$$\begin{array}{l} F^{\tau}\left(h, \tau_{\dot{h}=0}(h)\right) < 0 \text{ if } h < h^{ss} \\ F^{\tau}\left(h, \tau_{\dot{h}=0}(h)\right) = 0 \text{ if } h = h^{ss} \\ F^{\tau}\left(h, \tau_{\dot{h}=0}(h)\right) > 0 \text{ if } h > h^{ss} \end{array}$$

Note that $\lim_{\tau \to \left(\gamma \Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}} \frac{1}{h_h(h,\tau) - \frac{1-\gamma}{\gamma}\left(\frac{\tau(t)}{\Gamma}\right)^{\frac{1}{\gamma}}} = +\infty$. It follows from (76) that when $h < h^{ss}$, $\dot{\tau}(t) > 0$ for τ close enough to $\left(\gamma \Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}$. Thus, when $h < h^{ss}$ the locus $\dot{\tau}(t) = 0$ is in between $\tau_{\dot{h}=0}(h)$ and $\left(\gamma \Gamma^{\frac{1}{\gamma}} h\right)^{\frac{\gamma}{1-\gamma}}$. Furthermore when $\tau < \tau_{\dot{\tau}=0}(h)$ then $\dot{\tau}(h,\tau) < 0$ and if $\tau > \tau_{\dot{\tau}=0}(h)$ then $\dot{\tau}(h,\tau) > 0$. This implies that $\frac{\partial F^{\tau}(h, \tau_{\dot{\tau}=0}(h))}{\partial \tau} > 0$. ■

Now we proceed to prove proposition 12:

Proof. The dynamic system in a surrounding of the steady-state may be linearized as follows:

$$\begin{bmatrix} \dot{h}(t) \\ \dot{\tau}(t) \end{bmatrix} = \begin{bmatrix} \frac{\partial F_h(h^{ss}, s^{ss})}{\partial h} & \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau} \\ \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h} & \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau} \end{bmatrix} \begin{bmatrix} h(t) - h^{ss} \\ \tau(t) - \tau^{ss} \end{bmatrix}$$

The eigenvalues are as follows:

$$\begin{vmatrix} \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial h} - \lambda & \frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau} \\ \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h} & \frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau} - \lambda \end{vmatrix} = \lambda^2 - tr\,\lambda + Det = 0 \Rightarrow \lambda = \frac{tr \pm \sqrt{tr^2 - 4Det}}{2}$$

It follows from lemma 16:

$$\begin{aligned} tr &= \frac{\partial F_h\left(h^{ss}, \tau^{ss}\right)}{\partial h} + \frac{\partial F_\tau\left(h^{ss}, \tau^{ss}\right)}{\partial \tau} > 0 \\ Det &= \underbrace{\frac{\partial F_h\left(h^{ss}, \tau^{ss}\right)}{\partial h}}_{-} \underbrace{\frac{\partial F_\tau\left(h^{ss}, \tau^{ss}\right)}{\partial \tau}}_{+} - \underbrace{\frac{\partial F_h\left(h^{ss}, \tau^{ss}\right)}{\partial \tau}}_{+} \frac{\partial F_\tau\left(h^{ss}, \tau^{ss}\right)}{\partial h} = \\ Det &= \frac{\partial F_\tau\left(h^{ss}, \tau^{ss}\right)}{\partial \tau} \frac{\partial F_h\left(h^{ss}, \tau^{ss}\right)}{\partial \tau} \left[\frac{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial h}}{\frac{\partial F_h(h^{ss}, \tau^{ss})}{\partial \tau}} - \frac{\frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial h}}{\frac{\partial F_\tau(h^{ss}, \tau^{ss})}{\partial \tau}} \right] = \\ Det &= \underbrace{\frac{\partial F_\tau\left(h^{ss}, \tau^{ss}\right)}{\partial \tau}}_{+} \underbrace{\frac{\partial F_h\left(h^{ss}, \tau^{ss}\right)}{\partial \tau}}_{+} \underbrace{\left[\frac{\partial \tau_{\dot{\tau}=0}(h^{ss})}{\partial h} - \frac{\partial \tau_{\dot{h}=0}(h^{ss})}{\partial h} \right]}_{-} < 0 \end{aligned}$$

Thus, one of the eigenvalues is positive and the another is negative. This means that the steady-state is a saddle point. ■